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Inferior Factor in Cournot Oligopoly Revisited

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Inferior Factor in Cournot Oligopoly Revisited

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We reconsider the recent work by [Oku10] on (possibly asymmetric) Cournotian firms with two production factors, one of them being inferior. It is shown there that an increase in the price of the inferior factor does raise equilibrium industry output. In addition of providing a simpler and more rigorous proof of such a result, we generalize it to the case of technologies with $s \geq 2$ factors and allow some firms not to use the inferior one.

The article [Oku10] is interesting: (a) it is the first article that deals with an (homogeneous product) Cournot oligopoly with factor inferiority; (b) it uses an input-perspective point of view; and (c) a special method to solve for the Cournot equilibrium is adopted. In particular, in [Oku10] it is assumed that the technology of each firm can be described by a strictly concave production function with two production factors, one of them being inferior. In addition, it is also assumed that the individual marginal revenue of each firm is strictly decreasing. The main result in [Oku10] is that, in such a setting, the rise of the inferior factor price increases equilibrium industry output. However, in [Oku10, p. 131] we can read: 'However, we ought to consider more than two factors of production ... but getting definite comparative statistics results relating to factor inferiority seems to be impossible at this stage because of difficulty of deriving expressions corresponding to ... '.

In this note, we generalize the previous result to the case of $s \geq 2$ factors, also allowing some firm not to use the inferior one.[1] This is achieved by providing a simpler and more rigourous proof of the result in [Oku10], based on the characterization of an inferior factor in terms of the cost function properties (see for example [BR12]), and on an equilibrium uniqueness result of [MSS82] for Cournot oligopolies. In addition, we also provide a simple economic rationale for such a result.

Consider, in particular, a homogeneous Cournot oligopoly setting with n (≥ 1) firms. This can be interpreted as a game in strategic form with $N := \{1, \dots, n\}$ as player set, and with $\mathbb{R}_+$ as strategy set for firm, i.e. player, $i \in N$. The cost functions depend on the (positive) prices of s production factors. We only let vary the price w of one of the

[1] In [Oku10] it is assumed that each factor is used by any firm.

production factors in some open real interval W with $W \subseteq \mathbb{R}_{++}$.[2] The cost function of firm $i \in N$ now is a function $c_i : W \times \mathbb{R}_+ \to \mathbb{R}$ (with the first variable for the price). With price function $p : \mathbb{R}_+ \to \mathbb{R}$, the profit function of firm i is

$$f_i(w; x_1, \ldots, x_n) \ := \ p(\sum_{l=1}^{n} x_l)x_i - c_i(w; x_i).$$

We assume that $p > 0$ is twice continuously differentiable with

$$Dp < 0$$

and that the individual marginal revenue of each firm i is decreasing.[3] This condition is equivalent with concavity of the aggregate revenue function $r : \mathbb{R}_+ \to \mathbb{R}$, given by $r(y) \ := \ p(y)y$. Note that concavity of p is sufficient for the concavity of r. We assume that every c_i is twice continuously differentiable on[4] $W \times \mathbb{R}_+$ with

$$D_{xx}c_i > 0.$$

Thus every cost function $c_i(w, \cdot)$ is strictly convex.

The above situation guarantees ([MSS82, Lemma 5]) that for every $w \in W$ there exists *at most one* Cournot equilibrium, i.e. Nash equilibrium. In addition we now suppose for every $i \in N$ that

$$r(x) - c_i(w; x) \leq -c_i(w; 0) \ (w \in W)$$

for x large enough. This 'compactness' condition makes[5] that for every $w \in W$ there exists *at least one* Cournot equilibrium. Thus for every $w \in W$ there exists a *unique* Cournot equilibrium

$$\mathbf{e}(w) \ := \ (e_1(w), \ldots, e_n(w)).$$

Also we assume that $D_x c_i(w; 0) = 0$ ($i \in N, w \in W$), which implies that $\mathbf{e}(w) \in \mathbb{R}^n_{++}$, i.e. that each firm is active. Now let us bring in the assumption of equilibrium behaviour compatible with factor inferiority: assume for every $i \in N$ that each function $c_i(w; \cdot)$ is strictly increasing and that

$$D_{wx}c_i(w; e_i(w)) \leq 0 \ (w \in W) \tag{1}$$

such that in (1) the strict inequality holds for at least one firm. (1) corresponds to the assumptions in [Oku10] on the production functions for factor inferiority. Remember that by Shephard's Lemma, $D_w c_i(w; x_i)$ equals the so-called 'conditional' (cost-minimising) demand for the factor whose price is w expressed by firm i while producing x_i. So (1) says that the conditional demand of firm i for the factor actually decreases when its

[2] The interval W may be very small. It just should be such that (1) below holds.

[3] I.e. for each $k \in \mathbb{R}_+$ the function $r_k : \mathbb{R}_+ \to \mathbb{R}$ defined by $r_k(x) \ := \ p(x + k)x$ has a decreasing derivative.

[4] Note that the set $W \times \mathbb{R}_+$ is not open. So we mean here that c_i can be extended to an open subset of $\mathbb{R}^2_+$ containing $W \times \mathbb{R}_+$ on which it is twice continuously differentiable.

[5] A self-contained proof of this folklore result can be found in [vMQ12, Theorem 4].

output increases if and only if the marginal cost of firm i is decreasing with respect to w. For a discussion of this characterization see for example [BR12].

It follows that for each $w \in W$, $\mathbf{e}(w)$ is a solution of the system of n equations $-D_{x_i} f_i(w; \mathbf{x}) = 0$ $(i \in N)$, i.e. with $\underline{\mathbf{x}} := \sum_{l \in N} x_l$, of

$$-Dp(\underline{\mathbf{x}})x_i - p(\underline{\mathbf{x}}) + D_x c_i(w; x_i) = 0 \ (i \in N)$$

in the unknowns $x_1, \ldots, x_n > 0$. In the appendix we show that the implicit function theorem applies by showing that the Jacobi matrix of the mapping $G = (G_1, \ldots, G_n) : \mathbb{R}^n_{++} \times W \to \mathbb{R}^n$ defined by $G_i(\mathbf{x}; w) := -D_{x_i} f_i(w; \mathbf{x})$ is invertible. The implicit function theorem implies that each function $e_i : W \to \mathbb{R}$ is continuously differentiable. Let

$$\underline{\mathbf{e}} := \sum_{l \in N} e_l$$

be the equilibrium industry output. For all i and $w \in W$ we have

$$Dp(\underline{\mathbf{e}}(w))e_i(w) + p(\underline{\mathbf{e}}(w)) - D_x c_i(w; e_i(w)) = 0. \tag{2}$$

Differentiating (2) to w gives

$$(e_i D^2 p + Dp) D\underline{\mathbf{e}} + (Dp - D_{xx} c_i) De_i = D_{wx} c_i. \tag{3}$$

Note that (3) implies that $D\underline{\mathbf{e}}(w) \neq 0$ $(w \in W)$. We now prove that

$$D\underline{\mathbf{e}} > 0, \tag{4}$$

so that the equilibrium industry output is strictly increasing (on W), which is the main result in [Oku10].

As follows from (3), sufficient for (4) to hold is that

$$Dp(y) + xD^2 p(y) \leq 0 \ (x, y \in \mathbb{R}_+ \text{ with } x \leq y). \tag{5}$$

holds.[6] Indeed, if not then there is some $w \in W$ with $De(w) \leq 0$ and (3) and (1) imply $De_i(w) \leq 0$ $(i \in N)$ with at least one of these inequalities strict, which is impossible.

Sufficient for (5) to hold is that p is concave. We note that our assumptions imply that all best reply correspondences R_i are singleton-valued and that it is well-known that under condition (5) they are decreasing.

Showing that (4) also holds under our weaker assumption of concave aggregate revenue needs a more sophisticated reasoning which we give now. The proof is by contradiction. So fix $w \in W$ and assume $D\underline{\mathbf{e}}(w) < 0$. Let $N^- := \{l \in N \mid De_l(w) < 0\}$ and $\underline{\mathbf{e}}^-(w) := \sum_{l \in N^-} e_l(w)$. We have $\#N^- \geq 1$, $D\underline{\mathbf{e}}^-(w) < 0$ and $D\underline{\mathbf{e}}^-(w) \leq D\underline{\mathbf{e}}(w)$. (3) implies

$$\Big(\underline{\mathbf{e}}^-(w) D^2 p(\underline{\mathbf{e}}(w)) + (\#N^- + 1) Dp(\underline{\mathbf{e}}(w))\Big) D\underline{\mathbf{e}}(w) + Dp(\underline{\mathbf{e}}(w))(D\underline{\mathbf{e}}^-(w) - D\underline{\mathbf{e}}(w))$$

[6] It is easy to see that this condition is equivalent with $Dp(y) + yD^2 p(y) \leq 0$ $(y \in \mathbb{R}_+)$, which is in the oligopolistic literature called Novsheks' *marginal revenue condition* ([Nov85]).

$$-\sum_{l\in N^-} D_{xx}c_i(w;e_i(w))De_i(w) = \sum_{l\in N^-} D_{wx}c_i(w;e_i(w)).$$

As the left-hand side of this equality is positive and the right-hand side non-positive, we have a contradiction.

To provide an economic intuition for the previous formal result, by using (3) it is convenient to decompose De_i as it follows:

$$De_i = \frac{D_{wx}c_i}{e_iD^2p + 2Dp - D_{xx}c_i} + \frac{-\left(e_iD^2p + Dp\right)\sum_{l\neq i} De_l}{e_iD^2p + 2Dp - D_{xx}c_i}.$$

The first term equals D_wR_i: accordingly, the increase of the inferior factor price has the *direct effect* of shifting up the reaction curves of all firms which are using it. This implies an increase of production for those firms, for a *given* total output of the rivals. However, through the interaction among firms, there is also a *strategic effect*, captured by the second term in the decomposition. If the reaction curves of all firms are locally increasing (i.e., if $e_iD^2p+Dp > 0$, $i \in N$), the strategic effect is positive, and accordingly to (3) and (4), $De_i > 0$ *even if* $D_{wx}c_i = 0$. Conversely, if the reaction curve of firm i is locally decreasing (i.e., if $e_iD^2p+Dp < 0$), the strategic effect is negative if $\sum_{l\neq i} De_l > 0$ and it might more than compensate the direct effect (notice that by (3) and (4) that $De_i < 0$ *if* $D_{wx}c_i = 0$). Finally, if the reaction curve of firm i is locally flat (i.e., if $e_iD^2p + Dp = 0$), there is no strategic effect and $\text{sign}\,\{De_i\} = -\text{sign}\,\{D_{wx}c_i\}$.

Appendix

The Jacobi matrix of the mapping G is the the $n \times n$ matrix M given by

$$M_{ii} = -D_{x_ix_i}f_i(w;\mathbf{x}),\ \ M_{ij} = -D_{x_jx_i}f_i(w;\mathbf{x})\ \ (i \neq j).$$

Its determinant equals[7]

$$\prod_{l\in N}(D_{xx}c_l(w;x_l)-Dp(\underline{\mathbf{x}}))-\sum_{l\in N}\Big((Dp(\underline{\mathbf{x}})+x_lD^2p(\underline{\mathbf{x}}))\prod_{i\in N\setminus\{l\}}(D_{xx}c_i(w;x_i)-Dp(\underline{\mathbf{x}}))\Big). \quad (6)$$

We see that this determinant is non-zero if the marginal revenue condition (5) holds. However, it is not clear that it also is non-zero just under our assumption of concave aggregate revenue. Below we even show that it is positive.

In short notations, (6), can be rewritten as

$$\prod_{l\in N}(D_{xx}c_l - Dp)(1-\sum_{i\in N} k_i),\ \text{where}\ k_i = \frac{x_iD^2p(\underline{\mathbf{x}}) + Dp(\underline{\mathbf{x}})}{D_{xx}c_i(w;x_i) - Dp(\underline{\mathbf{x}})}.$$

So the determinant is positive if and only if $\sum_{i\in N} k_i < 1$. Let $K := \{i \in N \mid x_iD^2p + Dp > 0\}$. If $\#K = 0$, then again $\sum_{i\in N} k_i < 1$. Now suppose $\#K \geq 1$. This implies $D^2p > 0$. Let $m \in K$ be such that $D_{xx}c_m = \min_{i\in K} D_{xx}c_i$. Remembering that $\underline{\mathbf{x}}D^2p + 2Dp \leq 0$, it follows that

$$\sum_{i\in N} k_i \leq \sum_{i\in K} k_i \leq \sum_{i\in K}\frac{x_iD^2p + Dp}{D_{xx}c_m - Dp} = \frac{D^2p\sum_{i\in K} x_i + \#KDp}{D_{xx}c_m - Dp} \leq \frac{\underline{\mathbf{x}}D^2p + \#KDp}{D_{xx}c_m - Dp} < 1.$$

[7] See [OK71].

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