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Some Extensions of the class of K-matrices: A Survey and Some Economic Applications

Giorgio Giorgi
(Università di Pavia)

Cesare Zuccotti
(Università di Pavia)

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Via San Felice, 5
I-27100 Pavia
http://epmq.unipv.eu/site/home.html

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Some Extensions of the class of K-matrices: A Survey and Some Economic Applications

G. Giorgi[1] and C. Zuccotti[2]

Abstract

In this paper we take in to considerations some classes of matrices which are generalizations of the class of K-matrices, in the terminology of Fielder and Ptàk (1962). We study the hierarchy and inclusion relations between the classes considered and we point out some economic applications.

Key words

K-matrices; M-matrices; S-matrices; P-matrices; dominant diagonal matrices; stabiliy matrices; linear economic models.

1. Introduction

This paper takes into consideration some generalizations of the class of K-matrices (in the terminology of Fiedler and Ptàk (1962)). A (real) square matrix A of order n is called a Z-matrix or matrix of the Z-class if $a_{ij} \leqq 0$, $\forall i \neq j$. A Z-matrix is called a K-matrix if Z^{-1} is nonnegative (and obviously non zero, i. e. *semipositive*). There are many characterizations of the class of K-matrices: see the basic paper of Fiedler and Ptàk (1962) and the surveys of Poole and Boullion (1974), where the authors adopt the term "M-matrices" or better "nonsingular M-matrices", of Berman and Plemmons (1976), Plemmons (1977), Magnani and Meriggi (1981), Varga (1976a), Windisch (1989).

This paper is organized as follows.

Section 2 recalls the definitions of the classes considered in the present paper.

In Section 3 some known and new properties of the said classes are stated and other characterizations are established.

In Section 4 we provide the various inclusion and comparison results between the classes previously introduced and some further remarks are added.

In Section 5 some economic applications are pointed out.

[1] Faculty of Economics, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: ggiorgi@eco.unipv.it

[2] Faculty of Economics, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: czuccotti@eco.unipv.it

All matrices and vectors considered are real (the extension to the complex field is possible, even if the economic applications of this extension are often problematic). The notations $I, A^T, [0], rk(A)$ stand for: the identity matrix, the transpose of the matrix $A \in \mathbb{R}^{m\times n}$, the zero matrix, the rank of A. The notations $A > [0]$ and $A \geqq [0]$ are used to denote a *positive matrix* (all positive entries) and, respectively, a *nonnegative matrix* (all nonnegative entries). $A \geq [0]$ is used to denote a *semipositive matrix* (i. e. a nonzero, nonnegative matrix). Similar notations are used to denote positive, nonnegative and semipositive vectors. The notations $A \leqq [0]$, $A < [0]$, $A \leq [0]$ are obvious. The same convention is used also to compare two matrices (of the same order) or two vectors (of the same dimension). A_i denotes the $i-$th row of A, A^j the $j-$th column.

2. Classes of Matrices Considered

We consider the following classes of matrices $A \in \mathbb{R}^{m\times n}$.

(1) S-matrices The matrix A is an S-matrix if the system

$$\begin{cases} Ax > [0] \\ x \geqq [0] \end{cases}$$

admits a solution x.

(2) S_0-matrices The matrix A is an S_0-matrix if the system

$$\begin{cases} Ax \geqq [0] \\ x \geq [0] \end{cases}$$

admits a solution x.

(3) M-matrices The S_0-matrix A is an M-matrix (or an *irreducibly* S_0-matrix) if either A is an S_0-matrix and has only one column, or A is an S_0-matrix with $n > 1$ columns, but no matrix obtained from A by omitting at least one column, is an S_0-matrix. The terminology is due to Fiedler and Ptàk (1966), but this class must not be confused with the class of M-matrices of Poole and Boullion (1974), Berman and Plemmons (1976) and the other authors previously quoted.

(4) M_+-matrices The M-matrix A is an M_+-matrix if it admits a positive left inverse $A^{(+)}$ (in the usual sense of Moore-Penrose; see, e. g., Nashed (1976), Rao and Mitra (1971)):

$$A^{(+)}A = I, \quad A^{(+)} > [0] .$$

(5) M_0-matrices The M-matrix A is an M_0-matrix if and only if it is not an M_+-matrix.

We consider the following classes of matrices $A \in \mathbb{R}^{n\times n}$ (square matrices).

(6) P-matrices The square matrix A is a P-matrix if all its principal minors are positive.

(7) P_0-matrices The square matrix A is a P_0-matrix if all its principal minors are nonnegative.

(8) Q-matrices The square matrix A (*not necessarily symmetric*) is a Q-matrix if the following implication holds

$$x \neq [0] \Longrightarrow x^T A x > 0.$$

This class is also called the class of *quasi-positive definite matrices* (of positive definite matrices, when A is symmetric). This definition, at least in the economic literature, was considered by Samuelson (1947). Some authors (e. g. Nikaido (1968)) adopt the denomination "positive quasi-definite matrices".

(9) Q_0-matrices The square matrix A (*not necessarily symmetric*) is a Q_0-matrix if it is quasi-positive semidefinite, i. e.

$$x \neq [0] \Longrightarrow x^T A x \geqq 0$$

(obviously when A is symmetric, this class coincides withe the class of positive semidefinite matrices).

(10) DD-matrices (Dominant diagonal matrices) The square matrix A has a *row dominant diagonal* if there exist scalars $d_i > 0$, $i = 1, ..., n$, such that

$$d_i \mid a_{ii} \mid > \sum_{j \neq i}^{n} d_j \mid a_{ij} \mid, \quad i = 1, ..., n.$$

If, in addition, $a_{ii} > 0, i = 1, ..., n$, then A is said to have a *positive* row dominant diagonal. Similarly, if in addition, $a_{ii} < 0, i = 1, ..., n$, then A is said to have a *negative* row dominant diagonal. A has a *column dominant diagonal* if there exist $d_j > 0, j = 1, ..., n$, such that

$$d_j \mid a_{jj} \mid > \sum_{i \neq j}^{n} d_i \mid a_{ij} \mid, \quad j = 1, ..., n.$$

Similar definitions hold for positive or negative column dominant diagonal matrices. A fundamental property of DD-matrices is that if A has a row dominant diagonal, then A has a column dominant diagonal and vice-versa (Mc Kenzie (1960)). So, it is convenient to speak simply of DD-matrices.

(11) PS-matrices The square matrix A is *positive stable* if every its eigenvalue has a positive real part: $\text{Re}(\lambda_j) > 0, \forall j$. Usually, a square matrix A is said to be stable if it is *negative stable*, i. e. $\text{Re}(\lambda_j) < 0, \forall j$, as this condition is necessary and sufficient for the "global asymptotic stability" of the solutions $x(t)$ of a system of linear differential equations with constant coefficients

$$x'(t) = Ax(t),$$

with respect to the equilibrium solution $x^* = [0]$.

(12) PDS-matrices The square matrix A is *positive D-stable* if DA is PS, for every diagonal matrix $D \subset \mathcal{D}_{++}$, $\mathcal{D}_{++}$ being the class of diagonal matrices with *positive* diagonal elements.

(13) PTS-matrices The square matrix A is *positive totally stable* if every principal submatrix A_{JJ} of A is PDS.

We recall again the classes of the Z-matrices and of the K-matrices: a square matrix A is a Z-matrix if $a_{ij} \leqq 0, \forall i \neq j$; a Z-matrix A is a K-matrix if $A^{-1} \geq [0]$.

In what follows the letters S, S_0, etc. will also be used to denote the classes of the corresponding matrices: e. g. S is the class of all S-matrices of a fixed order (m, n). The same convention will be used for the transposed matrices (S^T, S_0^T,etc.).

3. Other Characterizations of the Classes Considered

◦ Characterizations of the S-matrices

The name of this class derives from the mathematician E. Stiemke who first introduced them (Stiemke (1915)). Most of the following characterizations are given by Fiedler and Ptàk (1966), some others are new.

(a) The system $Ax > [0]$ admits a solution $x \geqq [0]$.

(b) The system $Ax > [0]$ admits a solution $x \geq [0]$.

(c) The system $Ax > [0]$ admits a solution $x > [0]$.

(d) For any vector $p \geq [0]$, at least one component of the vector pA is positive.

(e) It holds the implication

$$p \geq [0] \Longrightarrow pA \nleq [0] .$$

(f) It holds the implication

$$pA \leqq [0] \Longrightarrow p \ngeq [0] .$$

(g) For any vector $p \leq [0]$, at least one component of pA is negative.

(h) It holds the implication

$$p \leq [0] \Longrightarrow pA \ngeq [0] .$$

(i) It holds the implication

$$pA \geqq [0] \Longrightarrow p \nleq [0] .$$

If A is *square*, then a characterization of S-matrices is related to the feasibility of the *linear complementarity problem* (LCP). See Cottle and others (1992). The LCP (q, A) is described as follows: given a vector $q \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$, find a nonnegative vector z such that

$$w = q + Az, \quad w \geqq [0] , wz = 0.$$

A vector $z \geqq [0]$ such that $q+Az \geqq [0]$ is said to be *feasible.* If also the complementarity condition $wz = 0$ holds, the vector z is called a *solution* of the LCP. The LCP (q, A) is said to be *solvable* if it has a solution. It can be proved the following result (Cottle and others (1992)).

Proposition 1 The matrix $A \in \mathbb{R}^{n\times n}$ is an S-matrix if and only if the LCP (q, A) is feasible, for all $q \in \mathbb{R}^n$.

◦ Characterizations of the S_0-matrices

Also this class is introduced by Fiedler and Ptàk (1966). The characterization (d) is, as far as we are aware, new.

(a) The system $Ax \geqq [0]$ admits a solution $x \geq [0]$.

(b) For any vector $p \geqq [0]$, at least one component of pA is nonnegative.

(c) It holds the implication

$$p \geqq [0] \Longrightarrow pA \not< [0] \, .$$

(d) $(A + B) \in S$, for any $B > [0]$.

The classes S and S_0 are obviously related by the inclusion $S \subset S_0$, but they are also related by the *Ville theorem of the alternative* (see, e. g., Cottle and others (1992)).

Let $A \in \mathbb{R}^{m\times n}$ be given. The system

$$Ax > [0] \, , x > [0]$$

has a solution if and only if the system

$$yA \leqq [0] \, , y \geq [0]$$

has no solution.

In terms of the classes S and S_0, the Ville theorem of the alternative can therefore be described by the following equivalence

$$(A \in S) \iff ((-A^T) \notin S_0).$$

Taking as a starting point two characterizations of P-matrices, given by Aganagić (1984) (see the next point 6, at letter (m)), Magnani (1984) has given the following two new characterizations of S-matrices and S_0-matrices.

- Let $A \in \mathbb{R}^{m\times n}$ and let $B \in \mathbb{R}^{m\times n}$, with $B_i \geq [0]$, $\forall i = 1, ..., m$. Then $A \in S_0$ if and only if $[(I - E)B + EA] \in S_0$, when $E \in \mathcal{D}_+$, $E \leqq I$, $\mathcal{D}_+$ being the class of diagonal matrices with nonnegative diagonal.

- Let $A \in \mathbb{R}^{m\times n}$ and let $B \in \mathbb{R}^{m\times n}$, with $B_i \geq [0]$, $\forall i = 1, ..., m$. Then $A \in S$ if and only if $[(I - E)B + EA] \in S$, when $E \in \mathcal{D}_+$, $E \leqq I$.

◦ Characterizations of the M-matrices

The following characterizations (a), (b) and (c) are due to Fiedler and Ptàk (1966); the characterization (d) can be obtained by means of Theorems 3.9, 3.10 and 3.11 in the same paper of Fiedler and Ptàk.

(a) Either $A \in S_0$ and A has only one column, or $A \in S_0$ but no matrix obtained from A by omitting at least one column is an S_0-matrix.

(b) $A \in S_0$ and the system

$$\begin{cases} Ax \geqq [0] \\ x \geq [0] \end{cases}$$

admits only solutions $x > [0]$.

(c) $A \in S_0$ and for any $x \neq [0]$, solution of $Ax \geqq [0]$, it holds either $x > [0]$ or $x < Ax = [0]$.

(d) $A \in S_0$ and, moreover, A verifies one of the following equivalent conditions.

(d_1) A admits generalized inverse $A^{(+)} > [0]$.

(d_2) For any vector $y \geq [0]$, there exists a solution $p > [0]$ of $pA = y$.

(d_3) $rk(A) = n$ and for every vector x such that $Ax \geq [0]$ it holds $x > [0]$.

(d_4) $rk(A) = n - 1$ and there exist vectors $p > [0]$ and $x > [0]$ such that $pA = [0]$, $Ax = [0]$.

(d_5) If $Ax \geqq [0]$ for $x \neq [0]$, then it holds either $Ax = [0] < x$ or $Ax = [0] > x$.

We state now some properties related to classes S, S_0, M, M_+ and M_0. Here, as before, A is a matrix with $m = m_A$ rows and $n = n_A$ columns.

(A) $A \in M_+ \Longrightarrow rk(A) = n_A$.
$A \in M_0 \Longrightarrow rk(A) = n_A - 1$.
$A \in M \Longrightarrow m_A \geqq n_A$.

(B) $A \in M_0 \Longleftrightarrow rk(A) = n_A - 1$ and the systems $Ax = [0]$ and $yA = [0]$ admit solutions $x > [0]$ and $y > [0]$.

(C) $A \in M_0 \Longrightarrow A \in M, \ -A \in S_0$.

(D) $A \in M_0$ if and only if $A \in S_0$ and the following implication holds:

$$Ax \geqq [0], x \neq [0] \Longrightarrow Ax = [0] \text{ and either } x > [0] \text{ or } x < [0].$$

(E) $M = M_+ \cup M_0$; $M^T = M_+^T \cup M_0^T$.

(F) $A \in M_+ \Longrightarrow A \in S^T$.

(G) $m = n \Longrightarrow M_+ = M_+^T$.

(H) $A \in M_0 \Longrightarrow A \notin S, \ A \notin S^T, \ A \in S_0, \ A \in S_0^T$.

(I) $m = n \Longrightarrow M_0 = M_0^T$.

The properties from (A) to (E) follow form the definitions of M, M_+, and M_0, taking into account Theorems 3.8 to 3.11 of Fiedler anf Ptàk (1966). Property F) stems directly from the definition of M_+, as soon as we note that if A admits a positive left-inverse $A^{(+)}$, then the system

$$yA = p, \quad p \geqq [0] \tag{1}$$

does not admit the solution

$$y = pA^{(+)} > [0]\,. \tag{2}$$

- Property (G) By definition of M_+ and a well-known property of the usual inverse A^{-1} of a square regular matrix A, any square M_+- matrix A admits an inverse $A^{-1} = A^{(+)} > [0]$. Therefore, $yA = [0]$ implies $y = [0]$ and each solution of (1) is positive, just as in (2). This shows that A^T is not only in S_0, but in M also. As $(A^T)^{-1} = (A^{(+)})^T > [0]$, A^T is in M_+, i. e. $A \in M_+^T$. The same arguments applied to a square M_+^T-matrix complete the proof of (G).

- Property (H) Let A be an M_0-matrix. Property (D) assures that A is out of S, whereas (C) shows, thanks to the Ville theorem of the alternative, applied to $(-A)$ instead of A, that A^T is out of S, i. e. $A \notin S^T$. Property (B) shows that A is in S_0^T also. Being M_0 a subclass of S_0, this completes the proof of (H).

- Property (I) This property trivially follows from (B) as soon as we note that, being A a square matrix, $rk(A) = n_A - 1$ if and only if $rk(A^T) = n_{A^T} - 1$ and the vectors $\bar{x} = y^T$ and $\bar{y} = x^T$, with x and y chosen as in property (B), play with A^T the same role played with A by x and y in the same property.

Finally, it may be useful to remark here also the properties:

(J) If $m = n$, then

$$\begin{cases} M_+ = M_+^T \\ A \in M_+ \text{ or } A \in M_+^T \Longrightarrow A \in S, A \in S^T. \end{cases}$$

(K) If $m = n$, then

$$\begin{cases} M_0 = M_0^T \\ A \in M_0 \text{ or } A \in M_0^T \Longrightarrow A \notin S, A \notin S^T, A \in S_0, A \in S_0^T \end{cases}$$

which follow from (F)-(G) and (H)-(I).

(L) If $m = n$, then the following two conditions are equivalent (Fiedler and Ptàk (1966)):

(i) A is nonsingular and $A \in M$;

(ii) $A^{-1}>[0]$.

This is a sort of "modified" monotonicity property, as, a square matrix A is usually called *monotone* or of *monotone kind* if

$$Ax \geqq [0] \Longrightarrow x \geqq [0]\,.$$

A basic result of Collatz (1966)states that the above implication is equivalent to: A^{-1} exists and it holds $A^{-1} \geq [0]$.

We now take into consideration the characterizations of the classes of *square* matrices previously considered.

◦ Characterizations of the P-matrices

This class was perhaps first considered by Ostrowski (1937) and by the economist John Hicks (1939); this last author, however, referred the characterization of a P-matrix, not to A but to $(-A)$: the square matrices whose principal minors of order k have the sign of $(-1)^k, k = 1, ..., n$, are called, in the economic literature, *Hicksian matrices* or also *NP-matrices.* As far as we know the name "P-matrix" was first given by Fiedler and Ptàk (1962) and by Gale and Nikaido (1965). Here we list the main characterizations of this class: see Fiedler and Ptàk (1966), Berman (1981), Plemmons (1977), Cottle and others (1992).

(a) Every principal minor of A is positive.

(b) For every vector $x \neq [0]$ there exists an index i such that $x_i(Ax)_i > 0$.

(c) The matrix A "reverses the sign" of the zero vector only, i. e.

$$x_i(Ax)_i \leqq 0 \Longrightarrow x = [0] \, .$$

(This characterization, obviously equivalent to (b), was first given by Gale and Nikaido (1965) in order to prove an important result on the global univalence of mappings).

(d) For every vector $x \neq [0]$ there exists in $\mathcal{D}_{++}$ (class of diagonal matrices with a positive diagonal) a matrix $D = D_x$, such that $x^T A D_x x > 0$.

(e) The same as (d), with a nonnegative diagonal replacing positive diagonal.

(f) The real eigenvalues of each principal submatrix of A (A included) is positive.

(g) The matrix $A + D$ is nonsingular, for each nonnegative diagonal matrix D.

(h) For each *signature matrix* S (i. e. a diagonal matrix with diagonal entries $s_{ii} = +1$ or -1), there exists an $x > [0]$ such that $SASx > [0]$.

The following characterization is particularly important in the theory of LCP and was given by Ingleton (1966), Cottle (1968), Tamir (1973). See also Cottle and others (1992).

(i) For every vector q, the LCP (q, A) has a *unique* solution.

The following characterization is due to Uekawa (1971) who applied the theory of P-matrices and other classes of matrices to the study of the Stolper-Samuelson theorem on factor price equalization. See also Uekawa and others (1973). Let us denote by J any subset of $N = \{1, ..., n\}$ ($J = N$ and $J = \varnothing$ are not excluded) and by D_J a diagonal matrix obtained from the identity matrix I by replacing each j-th row e^j by $-e^j$, $j \in J$. Uekawa (1971) gives the following characterization of P-matrices.

(l) An (n, n) real matrix A is a P-matrix if and only if, for any J, $\varnothing \subset J \subset N$, the inequality $x^T(D_J A D_J) > [0]$ has a solution $x^T > [0]$.

For other similar criteria working on $A \geqq [0]$, see Uekawa (1971) and Uekawa and others (1973).

(m) The following two characterizations have been given by Aganagić (1984).

(m_1) Let B arbitrarily chosen in $\mathcal{D}_{++}$, then $A \in P$ is and only if $\{(I - E)B + EA\} \in P$, when $E \in \mathcal{D}_+$, $E \leqq I$.

(m$_2$) Let B arbitrarily chosen in $\mathcal{D}_{++}$, then $A \in P$ is and only if $\{(I - E)B + EA\}$ is nonsingular, when $E \in \mathcal{D}_+$, $E \leqq I$.

We recall that the class of K-matrices is given by the intersection of the class of Z-matrices and the class of P-matrices:

$$K = Z \cap P.$$

∘ Characterizations of the P_0-matrices

The following characterizations are found in Fiedler and Ptàk (1966), except (g), due to Arrow (1974).

(a) The matrix A is a P_0-matrix, that is, all principal minor of A are nonnegative.

(b) For each vector $x \neq [0]$ there exists an index i such that $x_i \neq 0$ and $x_i(Ax)_i \geqq 0$.

(c) For any vector $x \neq [0]$ there exists a diagonal matrix $D = D_x \geqq [0]$ such that it holds $x^T D_x x > 0$, $x^T A^T D_x x \geqq 0$.

(d) All real eigenvalues of A, as well as of all its principal submatrices, are nonnegative.

(e) $(A + \varepsilon I) \in P$ for any $\varepsilon > 0$.

(f) $(A + D) \in P$, for any diagonal matrix $D \in \mathcal{D}_{++}$.

(g) For any diagonal matrix $D \in \mathcal{D}_{++}$ every real eigenvalue of DA is nonnegative.

Also P_0-matrices are important in the study of LCP. If in an LCP the vector $w = q+Az$ is constanmt for all solutions z, the solutions of such a problem are said $w-$*unique.* Cottle and others (1992) identify P_0-matrices as the class of matrices for which all solutions of the LCP must be w-*unique.*

∘ Characterizations of the Q-matrices

The notation for this class is, as far as we are aware, new. The following propertis are equivalent.

(a) A is quasi-positive definite, i. e.

$$x \neq [0] \Longrightarrow x^T Ax > 0.$$

(b) The symmetric matrix $(A + A^T)$ is positive definite.

(c) Every eigenvalue of $(A + A^T)$ is real and positive.

(d) Every principal minor of $(A + A^T)$ is positive, i. e. $(A + A^T) \in P$.

(e) Every *leading principal minor* (*or North-West principal minor*) of $(A + A^T)$ is positive.

The equivalence between (d) and (e) can be quickly proved as follows: $A \in Q \Longleftrightarrow (A+A^T)$ is positive definite $\Longleftrightarrow \Pi(A + A^T)\Pi^T$ is positive definite for any permutation matrix $\Pi \Longleftrightarrow$ all leading principal minors of $\Pi(A + A^T)\Pi^T$ are positive, for every permutation matrix $\Pi \Longleftrightarrow$ all principal minors of $(A + A^T)$ are positive, i. e. $(A + A^T) \in P$.

◦ Characterizations of the Q_0-matrices

From the characterizations of the class Q, the following characterizations of the class Q_0 are easily obtained.

(a) A is quasi-positive semidefinite, i. e.

$$x \neq [0] \Longrightarrow x^T A x \geqq 0.$$

(b) $(A + A^T)$ is positive semidefinite.

(c) Every eigenvalue of $(A + A^T)$ is real and nonnegative.

(d) Every principal minor of $(A + A^T)$ is nonnegative, i. e. $(A + A^T) \in P_0$.

Note that there is no characterization of Q_0 which is similar to (e) of the class Q.

◦ Characterizations of the DD-matrices (Dominant diagonal matrices)

DD-matrices and their generalizations have a long history, which goes back to Hadamard and other French mathematicians (see Marcus and Minc (1964). The definition 10) of Section 2 is due to the economist L. McKenzie (1960) and is largely ignored by mathematicians, also in recent contributions. See Giorgi and Zuccotti (2009), De Giuli, Magnani and Moglia (1994), for a survey on diagonal dominant matrices. Here we recall that a (real) square matrix A is said to have a *row* dominant diagonal, *in the sense of Hadamard* (RHDD) if

$$| a_{ii} | > \sum_{j \neq i} | a_{ij} |, \quad i = 1, ..., n,$$

(i. e. if in the Definition 10) of Section 2, it holds $d_i = 1, \forall i = 1, ..., n$).

A has a *column* dominant diagonal, in the sense of Hadamard (CHDD), if

$$| a_{jj} | > \sum_{i \neq j} | a_{ij} |, \quad j = 1, ..., n,$$

(i. e. if in the Definition 10) of Section 2 it holds $d_j = 1, \quad \forall j = 1, ..., n$).

It must be noted that the two properties (RHDD) and (CHDD) are *not* equivalent. Given the square matrix A, its *comparison matrix* $C = C(A)$ is given by

$$C = [c_{ij}], \quad c_{ij} = \begin{cases} \ \ | a_{ij} | & \text{if } i = j, \\ - | a_{ij} | & \text{if } i \neq j. \end{cases}$$

So, $C(A)$ is a Z-matrix. We denote by $\mathcal{D}$ the class of diagonal matrices and, as before, by $\mathcal{D}_{++}$ the class of diagonal matrices with a postitive diagonal.

The following propositions are equivalent.

(a) A is a row DD-matrix, i. e. there exist scalars $d_i > 0, \ i = 1, ..., n$, such that

$$d_i | a_{ii} | > \sum_{j \neq i} d_j | a_{ij} |, \quad i = 1, ..., n.$$

(b) A is a column DD-matrix, i. e. there exist scalars $d_j > 0,\ j = 1, ..., n$, such that

$$d_j \mid a_{jj} \mid > \sum_{i \neq j} d_i \mid a_{ij} \mid, \quad j = 1, ..., n.$$

(c) There exists a regular matrix $D \in \mathcal{D}$ such that $D^{-1}AD$ has a row dominant diagonal in the sense of Hadamard (RHDD).

(d) There exists a matrix $D \in \mathcal{D}_{++}$ such that for $D^{-1}AD$ the property sub (c) holds.

(e) There exists a regular matrix $D \in \mathcal{D}$ such that $D^{-1}AD$ has a column dominant diagonal in the sense of Hadamard (CHDD).

(f) There exists a matrix $D \in \mathcal{D}_{++}$ such that for $D^{-1}AD$ the property sub (e) holds.

(g) The comparison matrix $C(A)$ is a K-matrix.

(h) A has a dominant diagonal in the sense of Beauwens (1976) and Varga (1976b), i. e. with $C = [c_{ij}]$ the comparison matrix of A, the following system

$$\begin{cases} x > [0] \\ Cx \geq [0] \\ \sum_{j \leqq i} c_{ij} x_j > 0, \quad \forall i, \end{cases}$$

admits asolution.

(i) A^T has a dominant diagonal in the sense of Beauwens and Varga.

(l) There exist in $\mathcal{D}_{++}$ two matrices D and E such that, with $C = C(A)$ the comparison matrix of A, either the matrix $T = DCE$ has a dominant diagonal in the sense of Beauwens and Varga or T^T has the same property.

(m) A is of *generalized positive type,* in the sense of Varga (1976a), i. e. the system $C(A)x \geq [0]\,, x > [0]\,$, has solutions and, moreover, for each i such that $(C(A))_i x = 0$ there exists a chain connecting the index i with some index j and for which $(C(A))_j x > 0$.

(n) C^T is of generalized positive type in the sense of Varga.

Other extensions of the concept of dominant diagonal matrices are due to Pearce (1974) and Okuguchi (1978).

○ Characterizations of the PS-matrices

The followintg properties are equivalent.

(a) The square matrix A is positive stable, i. e. $\text{Re}(\lambda_j) > 0,\ \forall j$ (i. e. the real part of each eigenvalue of A is positive).

(b) There exists a symmetric positive definite matrix W such that

$$AW + WA^T$$

is positive definite (this is the famous Lyapunov criterion for the positive stability of A).

(c) $(A+I)$ is nonsingular and the matrix $G=(A+I)^{-1}(A-I)$ is convergent, i. e. $G^{((n)} \longrightarrow [0]$, for $n \longrightarrow +\infty$.

(d) $(A+I)$ is nonsingular, and for $G=(A+I)^{-1}(A-I)$ there exists a positive definite symmetric matrix W such that $W-G^TWG$ is positive definite.

For characterizations (c) and (d) see Plemmons (1977). Finally, we recall that a well-known basic result is the *algorithm of Routh and Hurwitz* (see Gantmacher (1959)) which yields necessary and sufficient conditions for the (negative) stability of A.

4. Other Inclusion and Comparison Results and Further Remarks

We have already discussed some properties related to the classes M, M_0 and M_+. We now point out other relations of inclusion, partial overlapping and disjunction between the classes of matrices considered.

(I) The inclusions

$$\begin{aligned} &S \subset S_0, \quad M \subset S_0, \quad S^T \subset S_0, \quad M^T \subset S_0^T, \\ &Q \subset P \subset S \end{aligned}$$

are all strict (for the case of S and S^T, S_0 and S_0^T, they hold either with $m=n$ or with $m \neq n$).

(II) The inclusions

$$Q_0 \subset P_0 \subset S_{0,} \quad Q \subset Q_0, \quad P \subset P_0$$

are all strict.

(III) The classes P_0 and Q_0 are not included in S. More precisely, P_0 and S have a partial overlapping (a nonempty intersection) and the same holds for Q_0 and S. Take, e. g., the following matrices

$$A = \begin{bmatrix} 10 & 7 \\ 10 & 7 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix}.$$

The first matrix is in P_0 and in S; the second matrix is in P_0, but not in S.

(IV) The classes P and Q_0 have a partial overlapping. Take, e. g., the following matrices

$$A = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}; \qquad B = \begin{bmatrix} 10 & 1 \\ 6 & 1 \end{bmatrix}.$$

The first matrix is in P and in Q_0; the second matrix is in P, but not in Q_0.

(V) If $m>n$, the inclusion $M_+ \subset S_0$ is strict, whereas the classes M_+ and S have a partial overlapping. The classes M_0 and S are disjoint, the classes M_0 and M_+ are disjoint. If $m=n$, the inclusions $M_+ \subset S \subset S_0$ are strict, the classes M_0 and S are disjoint, M_0 and M_+ are disjoint. If $m<n$, the inclusions $M_+ \subset S \subset S_0$ are strict.

(VI) We consider a DD-matrix with a *positive* diagonal: $a_{ii} > 0, \ \forall i = 1, ..., n$. We denote this class by PDD. The following inclusions are strict.

$$PDD \subset PS \text{ (class of positive stable matrices);} \quad PDD \subset P.$$

The first inclusion can be refined in the following way:

$$\begin{aligned} PDD &\subset \{\text{class of totally positive stable matrices}\} \subset \\ &\subset \{\text{class of positive } D\text{-stable matrices}\} \subset PS. \end{aligned}$$

(VII) The classes P and PS have only a partial overlapping:the class P is not included in PS and the class PS is not included in P. From a "historical" point of view, the fact that if A is negative stable, this does not imply that A is *Hicksian* $(-A \in P)$ and if A is Hicksian, this does not imply that A is negative stable, was pointed out, at least in the economic literature, in some fundamental results of Samuelson (1944, 1947). However, it must also be recalled a remarkable theorem of Fisher and Fuller (1968). Let $A \in P$; then there exists a positive diagonal matrix D such that $DA \in PS$ and all the characteristic roots of DA are simple.

(VIII) The classes Q and PDD have only a partial overlapping.

(IX) The inclusions

$$Q \subset PS; \quad Q \subset P$$

are strict. The first inclusion can be refined in the following way:

$$\begin{aligned} Q &\subset \{\text{class of totally positive stable matrices }\} \subset \\ &\subset \{\text{class of positive } D\text{-stable matrices}\} \subset PS. \end{aligned}$$

The second inclusion can be refined in the following way:

$$Q \subset \{\text{class of totally positive stable matrices}\} \subset P.$$

If $A \in Z$ (i. e. $a_{ij} \leqq 0, \ \forall i \neq j$), the following equalities hold

$$\begin{aligned} PS &= \{\text{class of totally positive stable matrices}\} = \\ &= \{\text{class of postive } D\text{-stable matrices}\} = P = PDD = K. \end{aligned}$$

Therefore, all the above conditions (before the last equality) are characterizations of the K-class. We recall again that K is the class of (square) Z-matrices with a semipositive inverse, or, equivalently, of class S, or, equivalently, of class P: $K = Z \cap S$, $K = Z \cap P$. More than 70 tests (!) are available to check whether a Z-matrix is in K. See, e. g., Berman and Plemmons (1976), Fiedler and Ptàk (1962), Poole and Boullion (1974), Plemmons (1977), Magnani and Meriggi (1981), Schröder (1978), Varga (1976a, b), Windisch (1989).

If A is *symmetric,* the following equalities hold

$$\begin{aligned} PS &= P = Q = \{\text{class of matrices with every leading principal minors positive}\} = \\ &= \{\text{class of positive definite matrices}\}. \end{aligned}$$

Moreover, the following equalities hold

$$P_0 = Q_0 = \{\text{class of positive semidefinite matrices}\} .$$

Note that, in the present case, the nonnegativity of the leading principal minors of A is not sufficient for A to in in Q_0. Take, e. g., the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

which has the two leading principal minors nonnegative (zero), but it is not positive semidefinite.

We make here some further remarks on the classes $S, S_0, M, P, P_0, Q, Q_0$. For the class PDD we refer the reader to the survey paper of Giorgi and Zuccotti (2009) and for the class PS we refer the reader to the paper of Giorgi (2003), where the usual negative stable matrices are examined.

• S-class

(1) If A has every row A_i semipositive ($A_i \geq [0]\,,\ \forall i = 1, ..., m$) or at least one column A^j positive ($\exists j : A^j > [0]$), then $A \in S$. If A has more than one column which is positive, then $A \notin M$.

(2) If $A \in Z$, then $A \in S$ if and only if $A \in K$.

(3) A matrix $A \in S$ if and only if for any vector $y \geq [0]$ at least one component of yA is positive (see Proposition 1 in Section 5).

• S_0-class

(1) If we denote by K_0 the class of square matrices A such that $(A + \varepsilon I) \in K$, for every $\varepsilon > 0$, we have the strict inclusion $K_0 \subset S_0$.

(2) If $A \in Z$, $\mid A \mid \neq 0$ and A is *indecomposable* (see, e. g. Debreu and Herstein (1953), Berman and Plemmons (1976)), then we have the implication

$$A \in S_0 \Longrightarrow A \in S, \quad A \in K.$$

Indeed, if $A \in S_0$, then there exists $x \geq [0]$ such that $Ax \geqq [0]$, and, thanks to regularity, $Ax \geq [0]$. This is sufficient, being A indecomposable, to have $A \in K$, and therefore $A \in S$.

• M-class

(1) If $A \in M$, then either $rk(A) = n$ or $rk(A) = n - 1$ (see Fiedler and Ptàk (1966), Proposition 3.6).

(2) If $A \in M$, then $m \geqq n$ (see Fiedler and Ptàk (1966), Proposition 3.8).

(3) If $A \in M$, A square, then also $A^T \in M$.

(4) If $A \in M$ and $rk(A) = n$, then $A \in S$.

(5) Let $A \in Z$. Then the following relations hold:

$M \subset K$, if $| A | \neq 0$;

$M = K$, if $| A | \neq 0$ and A is indecomposable;

$M \subset K_0$, if $| A | = 0$.

(6) If $A \in K_0$ and A is indecomposable, then there exists $x > [0]$ such that $Ax \geqq [0]$ (see Fiedler and Ptàk (1966), Proposition 5.8)

• P-class

(1) For *symmetric* matrices we have the equality (already remarked) $Q = P$, however it holds $P \neq S$: take, e. g. the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is in S, but not in P.

(2) If A is decomposable, then $A \in P$ if and only if it is a P-matrix every square block of its normal form, in the sense of Gantmacher (1959), containing the diagonal elements of A.

• P_0-class

(1) For *symmetric* matrices we have the equality $Q_0 = P_0$, however it holds $P_0 \neq S_0$: take, e. g., the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

which is in S_0, but not in P_0.

(2) If $A \in Z$, then $A \in K_0$ if and only if $A \in P_0$.

• Q-class

We have already remarked that, if A is *symmetric*, then $Q = P = PS$. Moreover, in this case, another characterization of the Q-class is:

(a) There exists a nonsingular matrix G such that $A = G^T G$.

• Q_0-class

We have already remarked that, if A is *symmetric*, then $Q_0 = P_0$. Moreover, in this case, other characterizations of Q_0-matrices are:

(a) All principal minors of A are nonnegative.

(b) All eigenvalues of A are nonnegative.

(c) There exists a matrix G such that $A = G^T G$.

(d) For every $\epsilon > 0$ we have $(A + \epsilon I) \in Q$.

Obviously, even for the symmetric case, the inclusion $Q \subset Q_0$ is strict.

5. Some Economic Applications

The classes of matrices considered in the previous sections have found several applications in a variety of fields: numerical analysis, linear complementarity problems, differential and difference equations, stochastic processes, economic models, problems of linear algebra, geometry, mathematical physics, etc. Here we shall be concerned only with some economic applications of the S-class and of the P-class. For economic applications of the K-class, of the DD-class and of the NS-class (negative stable matrices), the reader is referred to Nikaido (1968, 1972), Pasinetti (1977), Takayama (1985), Murata (1977), Woods (1978) and to the literature quoted in these books.

We consider a general linear economic model described by two nonnegative matrices:

- An input matrix $A \geqq [0]$, of order (m, n).
- An ouput matrix $B \geqq [0]$, of order (m, n).

Usually, due to the economic meaning of A and B, every column of A and B is required to be semipositive:

$$A^j \geq [0]\,, \quad B^j \geq [0]\,, \quad \forall j = 1, ..., n. \tag{3}$$

Also every row of B is required to be semipositive:

$$B_i \geq [0]\,, \quad \forall i = 1, ..., m. \tag{4}$$

This means that every good can be produced by some process; see, e. g., Kemeny, Morgenstern and Thompson (1956). The nonnegative column vector $x \in \mathbb{R}^n$ is the *activity vector,* therefore the quantities Bx and Ax describe, respectively, the gross productions and the inter-industry consumptions. The row vector $p \in \mathbb{R}^m$ (usually $p \geq [0]$) is the *price vector.* The vector $y = (B - A)x$ describes the *net productions,* obtained at the activity levels vector x, and the vector $v = p(B - A)$ describes the unitary net values, i. e. the values, at the price vector p, referred to the activity vector $x = u$, with $u^T = [1, 1, ..., 1]^T$. The model is *productive* if there exists an activity vector $x \geq [0]$ such that y is positive:

$$(B - A)x > [0]\,, \quad x \geq [0]\,.$$

We point out that, from a theoretical point of view, any arbitrary matrix can be written as a difference of two nonnegative matrices ("positive splitting"), but here A and B are given, they are the data of our economic model. The productivity of the model is therefore equivalent to the property

$$(B - A) \in S.$$

The model is *profitable* if there exists a price vector $p \geq [0]$ such that $v > [0]$:

$$p(B - A) > [0]\,, \quad p \geq [0]\,.$$

The profitability is therefore equivalent to the property

$$(B - A) \in S^T$$

i. e. $(B - A)^T \in S$. We have to remark that the two properties of productivity and profitability are compatible, but independent properties. In other words, the classes S and S^T are not disjoint, but have a partial overlapping. An exception is given by the case of A and B square, of order n, and $B = I$. So, $(B - A)$ becomes $(I - A) \in Z$ and, thanks to the closure of the K-class with respect to transposition, the model (A, I) is productive if and only if it is profitable. In the general case we can formulate the following test of productivity and profitability for a model (A, B).

Proposition 2 Let A and B be, respectively, the input and the output matrix of an economic linear model involving m goods and n processes. Then:

(i) The model (A, B) is productive if and only if, for any price vector $p \geq [0]$, there exists an activity (in general varying with the choice of p) such that the corresponding net value is positive:

$$p \geq [0] \Longrightarrow \exists j : p(B - A)^j > 0.$$

(ii) The model is profitable if and only if, for any activity vector $x \geq [0]$, there exists a good (in general varying with the choice of x) such that the corresponding net production is positive:

$$x \geq [0] \Longrightarrow \exists i : (B - A)_i x > 0.$$

We note that (i) is nothing but the characterization (d) of the S-matrices (see Section 3, point (1)). We give a complete proof of Proposition 2, for the reader's convenience.

Proof Thanks to the Ville theorem of the alternative (see Section 2), $(B - A) \in S$ if and only if $\left[-(B - A)^T\right] \notin S_0$. This means that the system

$$\begin{cases} \left[-(B - A)^T\right] p^T \geqq [0] \\ p^T \geq [0] \end{cases}$$

i. e. the system

$$\begin{cases} p(B - A) \leqq [0] \\ p \geq [0] \end{cases}$$

has no solution. Therefore (i) is proved. In a symmetric way, $(B - A)$ is profitable if and only if $(B - A)^T \in S$, i. e., thanks to the same theorem of the alternative, $\left[-(B - A)^T\right]^T \notin S_0$, i. e. $[-(B - A)] \notin S_0$.This means that the system

$$\begin{cases} -[(B - A)]\, x \geqq [0] \\ x \geq [0], \end{cases}$$

i. e. the system

$$\begin{cases} (B - A)x \leqq [0] \\ x \geq [0], \end{cases}$$

has no solution. Therefore (ii) is proved. □

Obviously, the practical relevance of the above tests relies on the possibility to detect non productive models and non profitable models, rather than productive models or profitable models.

If $(B - A) \in M$, $(B - A)$ *not square,* the number of processes is always less than the number of commodities $(n < m)$, thanks to Property 3.8 of Fiedler and Ptàk (1966). Moreover, if the columns of $(B - A)$ are linearly independent, then the pair (A, B) is profitable and *quasi-productive,* i. e. the system

$$\begin{cases} (B-A)x \geq [0] \\ x \geqq [0] \end{cases}$$

has a solution.

Mangasarian (1971) introduces the following assumption, in order to extend to the "matrix pencil", formed by the non necessarily square matrices A and B (and not necessarily nonnegative), the Perron-Frobenius theorem:

(a) $\quad A = BH, \ \ H \geqq [0]\,.$

This assumption does not assure, however, that the model (A, B) is productive, nor profitable. Mangasarian shows that a) is equivalent to the implication

$$pB \geqq [0] \Longrightarrow pA \geqq [0]\,,$$

of evident and acceptable economic meaning for $p \geqq [0]$, but with doubtful economic meaning if p has elements of opposite sign. We can introduce also the following variants of Mangasarian's assumption (a):

(b) $\quad A = HB, \ \ H \geqq [0]\,,$

(c) $\quad B = HA, \ \ H \geqq [0]\,,$

(d) $\quad B = AH, \ \ H \geqq [0]\,,$

for which the same above remarks hold. If we impose on the matrix H some further properties, then the Mangasarian's assumptions can assure productivity or profitability. Under the assumptions (3) and (4), if $\lambda^*(H) < 1$, $\lambda^*(H)$ being the Frobenius root of $H \geqq [0]$, then the model sub a) is productive, but not necessarily profitable; the model sub (b) is profitable, but not necessarily productive. The model sub (c) is profitable (but not necessarily productive) if H is indecomposable and $\lambda^*(H) > 1$; under the same assumptions on H^T, the model sub (d) is quasi-productive, but not necessarily profitable.

If we assume that A and B are *square* (as in the original Sraffa's joint production model; see, e. g., Schefold (1989)), then we can get more results and properties. Following Schefold (1989), the model is called "all-productive" if $(B - A)^{-1} \geq [0]$; the model is called "all-engaging" if $(B - A)^{-1} > [0]$. Obviously, if the model is all-productive, then it is also productive and profitable. The converse does not hold, as it can be shown by simple numerical examples, e. g. by choosing

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 3 & 6 \\ 2 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 8 & 5 \\ 4 & 3 & 5 \end{bmatrix}.$$

A classical result of Collatz (1966) shows that a square matrix A (not necessarily nonnegative) has a semipositive inverse $A^{-1} \geq [0]$, if and only if

$$Ax \geqq [0] \Longrightarrow x \geqq [0],$$

i. e. if A is of *monotone kind.* Therefore, we can say that the square model (A, B) is all-productive if and only if $(B - A)$ is of monotone kind. This property has an evident economic interpretation. Moreover, under the usual assumptions (3) and (4) on A and B, the pair (A, B) is all-productive if and only if there exists a semipositive and regular matrix C ($C \geq [0]$, $\mid C \mid \neq 0$), such that

$$x = Cy, \quad p = vC,$$

i. e. such that C transforms the vector of the net productions y in the activity vector x and the vector of the net values v in the price vector p.

Always under the assumption that A and B are square, thanks to a result of Fiedler and Ptàk (1966), we can assert that the following two conditions are equivalent:

(1) $(B - A)$ is regular and $(B - A) \in M$;

(2) $(B - A)^{-1} > [0]$, i. e. the model (A, B) is all-emgaging.

We remark that the model (A, B) is surely square if both $(B - A) \in M$ and $(B - A)^T \in M$: recall the quoted Property 3.8 of Fiedler and Ptàk (1966).

If $(B - A) \in Z$ (with A and B not necessarily both nonnegative), we can obtain, by imposing suitable properties on the matrices A and B which form the "splitting" $(B - A)$, some mathematical results, useful for the analysis of the linear joint production models we are discussing.

Proposition 3 Let $C \in Z$; then $C \in K$ if and only if C admits the splitting

$$C = B - A$$

with A and/or B in the S-class, B regular, $B^{-1}A \geqq [0]$ and $\lambda^*(B^{-1}A) < 1$.

Proof First let us assume that $C \in K$. One of the characterizations of the K-class is: the system

$$Cx > [0], \quad x \geqq [0]$$

admits solution, i. e. $C \in S$. But then, by choosing $A = [0]$ we have that $B = C$ and therefore $B \in K$, which entails $\mid B \mid \neq 0$, $B^{-1}A = [0]$, $\lambda^*(B^{-1}A) = 0$. Now we prove the converse implication. Let C verify the assumptions of the Proposition; then $B^{-1}A \geqq [0]$. We have $B^{-1}C = B^{-1}(B - A) = (I - B^{-1}A) \in Z$. Being $\lambda^*(B^{-1}A) < 1$, it holds $B^{-1}A \in K$. Therefore, $(I - B^{-1}A)^{-1} \geq [0]$, i. e. $(B^{-1}C)^{-1} \geq [0]$, i. e. $C^{-1}B \geq [0]$ (therefore C is regular). If $B \in S$, then $\exists q \geqq [0]$ such that $Bq > [0]$, i. e. $CC^{-1}Bq > [0]$, i. e. $C(C^{-1}Bq) > [0]$. But, being $q \geqq [0]$ and $(C^{-1}B) \geqq [0]$, we have $\bar{q} = C^{-1}Bq \geqq [0]$. Therefore $\exists \bar{q} \geqq [0]$ such that $C\bar{q} > [0]$, that is $C \in S$. If $A \in S$, then there exists $x \geqq [0]$ such that $Ax > [0]$, i. e. such that $C\left[(C^{-1}B)(B^{-1}A)\right]x > [0]$. Therefore, with

$\bar{x} = (C^{-1}B)(B^{-1}A)x$, being $x \geqq [0]$, $C^{-1}B \geqq [0]$ and $B^{-1}A \geqq [0]$, we have that $\exists \bar{x} \geqq [0]$ such that $C\bar{x} > [0]$, i. e. $C \in K$. □

Obviously, if in the above proposition, $A \geqq [0]$, $B \geqq [0]$, the thesis assures that the model (A, B) is all-productive. However, from a purely economic point of view, the case $(B - A) \in Z$ is of scarse interest, as it implies that we have only a "formal" joint production model. See Peris and Villar (1993). For other considerations on the splitting of Z-matrices, see Price (1968) and Varga (1962).

The economic meaning of the implications (A and B square):

$$((B-A) \text{ quasi-positive definite [i. e. } (B-A) \in Q]) \Longrightarrow$$
$$\Longrightarrow (B-A) \in P \Longrightarrow ((B-A) \text{ is productive and profitable}$$

is reduced, unless $(B - A) \in Z$. Indeed, in this case we have

$$(B-A) \in Q \Longrightarrow (B-A) \in P \Longleftrightarrow (B-A) \in K \Longleftrightarrow$$
$$\Longleftrightarrow ((B-A) \text{ is productive}) \Longleftrightarrow ((B-A) \text{ is profitable}).$$

We remark that the inclusion $Q \subset P$ is strict, even in the case of Z-matrices; take, e. g. the Z-matrix

$$A = \begin{bmatrix} 10 & -1 \\ -6 & 1 \end{bmatrix}$$

which is in P, but not in Q.

As for what concerns economic applications of the P-matrices, perhaps the most quoted appications are given by the *Hawkins-Simon conditions,* for the matrix $(I-A) \in Z$, where A is a *Leontief matrix or input-output matrix* (see Hawkins and Simon (1949), Nikaido (1968, 1972): the Hawkins-Simon conditions simply require that $(I - A) \in P$, which is equivalent, being $(I - A) \in Z$, that all the *leading principal minors* of $(I - A)$ are positive.

Moreover, the concept of a P-matrix, besides being useful in the analysis of input-output Leontief models and also in proving the Perron-Frobenius theorem, finds applications in the stability analysis of a Walrasian model of economic equilibrium. Indeed, Hicks' "perfect stability conditions" (Hicks (1939)) were given in terms of the signs, alternatively negative and positive, of the principal minors of the Jacobian matrix of excess demand functions: in other words, the negative of this Jacobian matrix belongs to P. It is merit of P. A. Samuelson (1941, 1944, 1947) to have shown that the Hicksin conditions are totally unrelated to a "true" dynamic stability, stemming from a system of differential equations. Recall that $P \not\Rightarrow PS$ and $PS \not\Rightarrow P$. Samuelson, in the quoted papers, stated, however, some conditions on the Jacobian matrix A, in order that the Hicksian conditions imply negative stability: (i) A is quasi-negative definite; (ii) A is symmetric.

L. Metzler (1945) recognized that if $(-A) \in Z$, then A is negative stable if and only if the Hicks' condition hold. In the economic literature, a square matrix A, such that

$(-A) \in Z$, is called "Metzlerian". This assumption has importance also in the analysis of *global stability* of a Walrasian economic system (see, e. g., Arrow and Hahn (1971), Karlin (1959), Nikaido (1968), Takayama (1985)).

Finally, we point out some important economic consequences of the famous *Gale-Nikaido theorem* on global univalence of mappings. See Gale and Nikaido (1965), Nikaido (1968) and the important overview on this subject made by Parthasarathy (1983). It is well-known (see, e. g., Apostol (1957)) the following local univalence theorem.

Local Univalence Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuously differentiable on the open set $S \subset \mathbb{R}^n$. Let $T = f(S)$. Denote the Jacobian of f at x by $\nabla f(x)$. Suppose that $| \nabla f(x) | \neq 0$ for some $x^0 \in S$. Then there exists a uniquely determined function g and two open sets $X \subset S$, $Y \subset T$, such that

(i) $x^0 \in X$, $f(x^0) \in Y$;

(ii) $Y = f(X)$;

(iii) f is one to one on X;

(iv) g is defined on Y; $g(Y) = X$, $g(f(x)) = x$, $\forall x \in X$.

(v) g is continuously differentaible on Y.

The first general extension of this theorem to the global case is due to Gale and Nikaido (1965), even if some basic, but not wholly correct, intuitions were anticipated by Samuelson.

Global Univalence Theorem of Gale and Nikaido Let $f : X \longrightarrow \mathbb{R}^n$ be a differentiable function on a rectangular region $X \subset \mathbb{R}^n$, i. e. $X = \{x \in \mathbb{R}^n : p_i \leqq x_i \leqq q_i\}$ (here p_i, q_i are real numbers where we may allow some or all of them to assume $-\infty$ or $+\infty$). If the Jacobian matrix $\nabla f(x)$ is a P-matrix for all $x \in X$, then f is global univalent on X.

Another version of the Global Univalence Theorem is due to Inada (1971), under continuous differentiability assumptions. If $X \subset \mathbb{R}^n$ is a convex set, it is possible to obtain another version of the Global Univalence Theorem; also this version is due to Gale and Nikaido.

Second Version of the Global Univalence Theorem of Gale and Nikaido Let $X \subset \mathbb{R}^n$ be a convex set and $f : X \longrightarrow \mathbb{R}^n$ be differentiable on X; if either the Jacobian matrix $\nabla f(x)$ is a Q-matrix on X or $-\nabla f(x)$ is a Q-matrix on X, then f is global univalent on X.

The Global Univalence Theorem of Gale and Nikaido has been generalized by various authors. A significant extension is due to Garcia and Zangwill (1979).

Theorems about global univalence are useful in several economic applications. For example, in establishing the "factor price equalization" in the theory of international trade (see, e. g., Chipman (1969), Inada (1971), Stolper and Samuelson (1941), Uekawa (1971), Uekawa and others (1973)). We quote from Sydsaeter and others (2008): "Suppose that a national economy has n different industries each producing a positive amount of a single output under constant return to scale, using other goods and scarce primary factors as inputs. Suppose the country is small and faces a fixed price vector p in $\mathbb{R}^n_+$ at which it can import or export the n goods it produces. Suppose there are n primary factors whose prices are given by the vector $w \in \mathbb{R}^n_+$. Equilibrium requires that $p_i = c_i(w)$ for each $i = 1, ..., n$, where $c_i(w)$ is the minimum cost at prices w producing one unit of good i. Then the vector equation $p = c(w)$, if it has a unique solution, will determine the factor price vector w as a function of p. When different countries have the same unit cost functions, this implies *factor price equalization* - because p is the same for all countries that trade freely, so is the factor price w."

Another economic application of the Global Univalence Theorem, and hence of the P-matrices, is in obtaining the uniqueness of the equilibrium price vector in a general walrasian model of pure exchange (see Arrow and Hahn (1971), Nikaido (1968)). We are given n single-valued functions $E_i(p)$, $i = 1, ..., n$, defined in a common domain P. $E_i(p)$ stands for the amount of excess demand for the $i-$th good. The behaviour of these n functions represents the state of an economy involving n goods. Usually, the following basic assumptions are made on $E_i(p)$:

(i) $P \subset \mathbb{R}^n_+$, $0 \notin P$, $P \neq \varnothing$, P open, and $\lambda P \in P$ whenever $p \in P$ and $\lambda > 0$.

(ii) Homogeinity of degree zero in p, i. e. $E_i(\lambda p) = E_i(p)$ for any $\lambda > 0,\ p \in P$ $(i = 1, ..., n)$.

(iii) The *Walras law in the narrow sense* holds, that is

$$\sum_{i=1}^{n} p_i E_i(p) = 0, \text{ for all } p \in P.$$

We define a price vector $\hat{p} = (\hat{p}_i)$ to be an *equilibrium price vector* of a system of excess demand functions $E_i(p)$ on P if $\hat{p} \in P$ and $E_i(\hat{p}) \leqq 0$ for all $i = 1, ..., n$, $E_i(\hat{p}) = 0$ whenever $\hat{p}_i > 0$. If the $n-$th good is taken as the "numeraire", the determination of $\hat{p}$ can be reduced to the study of the following equations and inequalities in the unknowns $p_1, ..., p_{n-1}$:

$p_i \geqq 0, \quad i = 1, ..., n-1;$

$E_i(p_1, ..., p_{n-1}, 1) \leqq 0, \quad i = 1, ..., n-1;$

$\sum_{i=1}^{n-1} p_i E_i(p_1, ..., p_{n-1}, 1) = 0.$

It is possible to show that the above system has a unique solution under the assumption that the Jacobian matrix J of order $n-1$, formed by the gradients $\nabla E_i(p_1, p_2, ..., p_{n-1})$, $i = 1, ..., n-1$, is *Hicksian* (i. e. $-J$ is a P-matrix) on a rectangular region of prices in P (see Nikaido (1968)).

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