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Fabio Tramontana
(Università di Pavia)

Viktor Avrutin
(Università di Urbino)

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I-27100 Pavia
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Complex endogenous dynamics in a one-sector growth model with differential savings

Fabio Tramontana[*1] **and Viktor Avrutin**[§]

[*] Department of Economics and Management, University of Pavia, Italy.
[§] DESP, University of Urbino "Carlo Bo", Urbino, Italy,
and IST, University of Stuttgart, Stuttgart, Germany.

Abstract.

We show that cyclic and chaotic dynamics may emerge in a Kaldor-Pasinetti growth model with different saving propensities, Leontief technology and logistic labor force growth rate.

1 Introduction

Neoclassical one-sector models of economic growth such as the one introduced by Ramsey (1928) or the so-called Solow-Swan model (Solow, 1956, Swan, 1956) predict that capital and output per capita converge to a steady state through a growth path. In order to obtain cycles or even chaotic dynamics we need to relax some assumptions or to introduce some additional characteristics into the models. To our knowledge, Richard Day has been the first researcher to look for such conditions. In a couple of papers dating back to the 80s (Day, 1982, 1983) a discrete-time version of the Solow model with a production lag has been considered. Day obtains a first-order difference equation, i.e. a one-dimensional map, describing the dynamics of the capital-labor ratio. By introducing, for instance, a productivity inhibiting effect or a variable savings ratio, he gets a one-dimensional map defined by a function that is concave and single humped, leading to possible irregular growth cycles.

In the so-called Kaldor-Pasinetti model (Kaldor, 1956, 1957, Pasinetti, 1962) two income groups are introduced, endowed with different saving propensities. Now, the aggregate saving propensity may vary according to the income distribution between the two groups, and this may have the same effects caused by the saving functions used by Day. Böhm and Kaas (2000) and Tramontana et al. (2011) move from a one-sector growth model in the spirit of Kaldor and Pasinetti (i.e. with two different but constant saving propensities) and show that by using some kind of production function such as the Leontief one, endogenous cycles may arise. In these cases, the one-dimensional map governing the dynamics of the capital-labor ratio is defined by a piecewise-linear function, so the source of endogenous fluctuations is not a nonlinearity in the function as in the case of Day, but the introduction of a discontinuity due to the assumption of Leontief technology.

In this branch of the economic growth literature, it is usually assumed that the labor force growth rate is constant, following the Malthusian growth model.

This assumption implies an unbounded size of the labor force, without any saturation level. In order to remove this too simplistic assumption it is possible to adopt a logistic growth of the labor force (Verhulst, 1938). Recently, some researchers (see Brida and Accinelli, 2007, Guerrini 2006, 2010a,b,c,d) have analyzed the consequences of a logistic growth rate of labor force in a Ramsey model and a Solow-Swan model, focusing on transitional dynamics.

In this paper we analyze the effects of a logistic growth rate of labor force in a Kaldor-Pasinetti model in the Böhm and Kaas (2000) version. In particular, we show the joint effect of the discontinuity (consequence of the Leontief technology) and the nonlinearity (due to the logistic equation) in the arising of complicated capital dynamics.

[1]Corresponding Author: University of Pavia, Department of Economics and Management, Via S.Felice 5, 27100 Pavia (PV), Italy. email: fabio.tramontana@unipv.it

2 Setup of the model

We consider a standard neoclassical one-sector growth model with workers and shareholders in the spirit of Kaldor (1956, 1957) and Pasinetti (1962). These groups of agents are characterized by different but constant saving propensities, s_w and s_r respectively, with $0 \leq s_w < s_r \leq 1$.

As usual, the wage rate w is determined as follows:

$$w(k) = f(k) - kf'(k)$$

where k denotes capital per worker and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a production function.

The marginal product of capital $f'(k)$ is received by shareholders, implying that total capital income per worker is $kf'(k)$.

The dynamics equation describing the accumulation of capital is the following:

$$k_{t+1} = G(k_t) := \frac{1}{1+n_t}\left[(1-\delta)\,k_t + s_w w(k_t) + s_r k_t f'(k_t)\right] \tag{1}$$

where $0 < \delta \leq 1$ is the capital depreciation rate and n_t is the labor force growth rate at time t. By considering $s_r = s_w$ and $n_t = n$ (i.e. constant) we obtain the standard Solow (1956) growth model.

Following Böhm and Kaas (2000) we use the so-called Leontief production function defined as follows:

$$f(k) = \min(ak, b) + c, \qquad a, b, c > 0 \tag{2}$$

originating the discontinuous map:

$$k' = G(k) := \begin{cases} G_L(k) = \frac{1}{1+n}\left[(1-\delta+s_r a)\,k + s_w c\right] & \text{if} \quad k \leq \frac{b}{a} \\ \\ G_H(k) = \frac{1}{1+n}\left[(1-\delta)\,k + s_w(b+c)\right] & \text{if} \quad k > \frac{b}{a} \end{cases} \tag{3}$$

where "$'$" is the unit-time advancement operator.

The dynamics of map (3) with constant labor force growth rate has been studied by Böhm and Kaas (2000) and Tramontana et al. (2011). They prove how easily growth cycles may endogenously arise in this setting.

We think that the exponential growth of the labor force is a limit of this growth model and consider more realistic the assumption of a logistic equation regulating the labor force growth rate. With this assumption we obtain the following two-dimensional discontinuous map having a so-called triangular structure[2]:

$$\begin{aligned} n' &= \mu n(1-n) \\ \\ k' &= \begin{cases} G_L(k) = \frac{1}{1+n}\left[(1-\delta+s_r a)\,k + s_w c\right] & \text{if} \quad k \leq \frac{b}{a} \\ \\ G_H(k) = \frac{1}{1+n}\left[(1-\delta)\,k + s_w(b+c)\right] & \text{if} \quad k > \frac{b}{a} \end{cases} \end{aligned} \tag{4}$$

with $1 < \mu < 4$.

3 Convergence to a steady state

Let us here consider the region of the parameters' space that leads to convergence to a steady state. It can be easily found that the map (4) admits at most two steady states, that we denote L and H[3]. The steady states are defined as follows:

$$\begin{aligned} L &: (n, k) = (n^*, k_L^*) \\ \\ H &: (n, k) = (n^*, k_H^*) \end{aligned} \tag{5}$$

[2] A trangular map has the following structure: $(x', y') = (f(x), g(x, y))$.

[3] We exclude from the analysis the steady states associated with $n = 0$ because they are not interesting. Moreover they are locally unstable under our parameter restrictions.

where

$$k_L^* = \frac{s_w c}{n^* + \delta - s_r a}, \quad k_H^* = \frac{s_w(b+c)}{n^* + \delta}, \quad n^* = 1 - \frac{1}{\mu} \tag{6}$$

Obviously the steady states really exist only if they belong to their definition regions. We can state the following:

Proposition 1 *The map (4) admits two steady states provided that $C1$ and $C2$ are both satisfied, where:*

$$C1 : s_w \leq \frac{b(n^* + \delta - s_r a)}{ac}, \quad n^* + \delta - s_r a > 0$$

$$C2 : s_w > \frac{b(n^* + \delta)}{a(b+c)}$$

If only $C1$ or $C2$ is fulfilled then only one steady state exists. If both conditions are violated then the map has no steady.

Proof. *The conditions follow directly by imposing that the steady states belong to their definition regions. In fact, $k_L^* \leq \frac{b}{a}$ iff $C1$ is fulfilled, while $k_H^* > \frac{b}{a}$ iff $C2$ holds.* ■

For the local stability of the steady states we have the following result:

Proposition 2 *If L and H exist, they are also locally stable provided that condition $C3$ holds, where:*

$$C3 : 1 < \mu < 3$$

Proof. *The eigenvalues of map (4) evaluated at the steady states are $(\lambda_n, \lambda_{kL})$ for L and $(\lambda_n, \lambda_{kH})$ for H, where:*

$$\lambda_n = 2 - \mu, \quad \lambda_{kL} = \tfrac{1-\delta+s_r a}{1+n^*}, \quad \lambda_{kH} = \tfrac{1-\delta}{1+n^*}$$

If $C1$ holds then $0 < \lambda_{kL} < 1$, while if $C2$ holds it is $0 < \lambda_{kR} < 1$. In order to have $|\lambda_n| < 1$ we need $1 < \mu < 3$, that is condition $C3$. ■

Summarizing, if at least one between $C1$ and $C2$ holds and the labor force growth rate converges to a steady state (i.e. $C3$ is fulfilled) then the capital per worker will converge either to k_L^* or to k_H^*.

4 Growth cycles and chaotic growth

The main aim of this paper is to show what happens when the conditions for the convergence to a steady stare are not fulfilled.

We distinguish among three different scenarios, according to the cause originating them:

1. Logistic cycles and chaos
2. Discontinuity induced growth cycles
3. Mixed cases

By explaining these scenarios we make use of a two-dimensional bifurcation diagram shown in Fig.1, and it is also possible to interpret some of the bifurcations leading to the structure there observed.

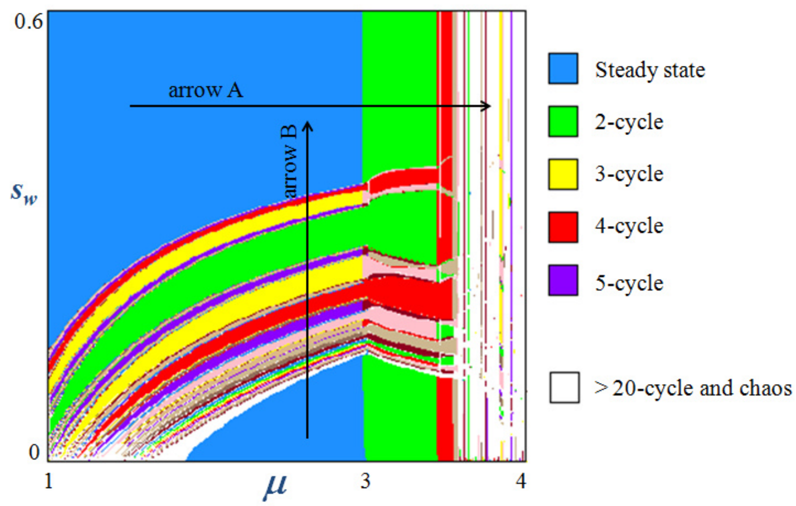

Figure 1: Two-dimensional bifurcation diagram with μ on the horizontal axis and s_w on the vertical one. The light blue color denotes the regions where conditions C1, C2 and C3 hold simultaneously. Outside this region growth cycles or chaotic growth occur. Parameters: $s_r = 0.6$, $a = 1.5$, $b = c = 2.9$ and $\delta = 0.45$.

4.1 Logistic cycles and chaos

Let us consider a combination of parameters such that conditions $C1$ and $C2$ hold. In this case by increasing the parameter μ of the logistic equation regulating the growth rate of the labor force, we can see the typical period-doubling route to chaos (Fig. 2a). We must take into account that our map (4) is triangular, and period cycles or chaos in the labor force growth rate, are transmitted to the capital per worker. So, we have growth cycles or a chaotic growth due to the instability in the labor force growth rate.

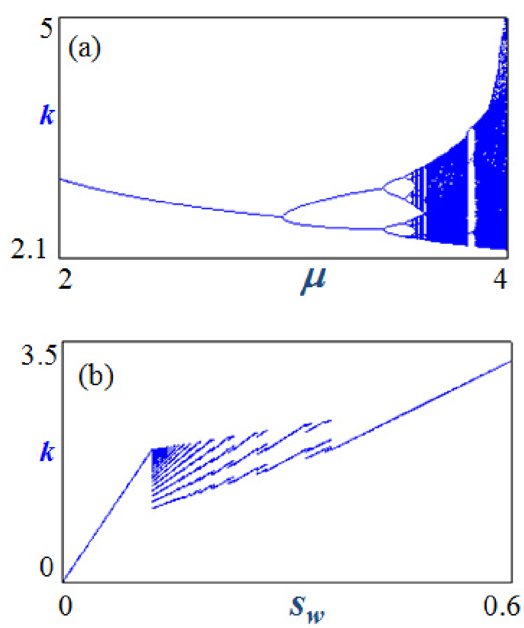

Figure 2: (a) period-doubling route to chaos caused by an increasing of the logistic parameter μ along the arrow A in Fig. 1. (b) period-adding structure characterizing growth cycles caused by the discontinuity of the map, obtained by varying s_w along the arrow B in Fig. 1. Parameters: $s_r = 0.6$, $a = 1.5$, $b = c = 2.9$, $\delta = 0.45$ with $s_w = 0.5$ in (a) and $\mu = 2.7$ in (b).

4.2 Discontinuity induced growth cycles

Growth cycles are possible also when the labor force growth rate is low enough to ensure a convergence to a positive value of n. This is the case studied by Böhm and Kaas (2000) and Tramontana et al. (2011). They show that when both conditions $C1$ and $C2$ are violated growth cycles of any period may arise, organized according to the so-called *period adding structure* and caused by *border collision bifurcations* (see Gardini et al. 2010, Avrutin et al. 2010). In Fig. 2b we have an example of how cycles with different periods are organized by varying the value of the saving propensity of workers.

One of the main characteristics of this structure is that coexistence of stable attractors (*multistability*) is not possible.

4.3 Mixed cases

The most interesting scenario is the one in which conditions $C1$, $C2$ and $C3$ are all violated. This case is not only characterized by the complicated dynamics shown in the previous cases, but also may multistability may occur. In Fig. 3 we show an example with parameters in this region, and in the phase space there is coexistence of two attracting cycles of period 16. So the outcome if the dynamics becomes path dependent and some exogenous shocks hitting the capital per worker can easily lead to a change in the attractor of the dynamical system.

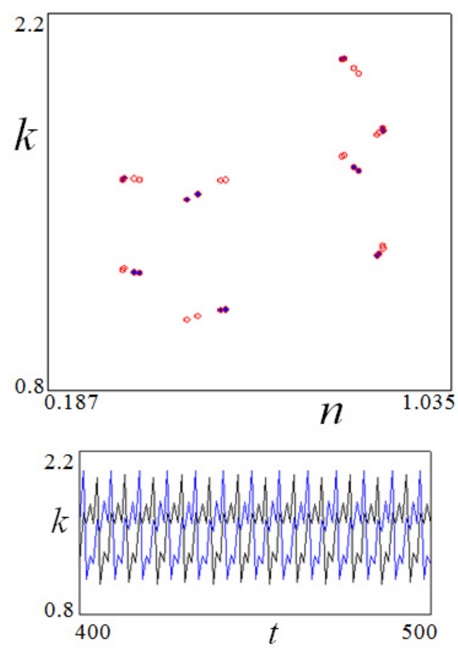

Figure 3: coexisting cycles in the phase plane. One cycle is denoted by red points, the other by red circles with white interiors. Below the corresponding timeplots. Parameters: $s_r = 0.6$, $a = 1.5$, $b = c = 2.9$, $\delta = 0.45$, $s_w = 0.15$ and $\mu = 3.568$.

Moreover, the routes to chaos are now modified with respect to the previous cases as testified by the bifurcation diagrams in Fig. 4, where the typical period-doubling cascade governing the dynamics of the labor force growth rate is compared to the route to chaos involving the capital per worker.

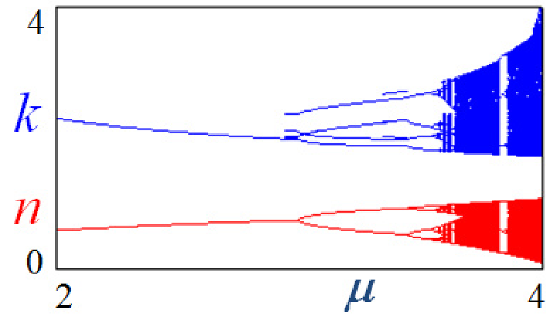

Figure 4: The period-doubling route to chaos characterizing n is compared with the bifurcations occurring to the capital per worker k. Parameters: $s_r = 0.6$, $a = 1.5$, $b = c = 2.9$, $\delta = 0.45$ and $s_w = 0.37$.

Finally, Fig. 5 shows a typical chaotic attractor occurring in this region, together with the corresponding chaotic time evolution of k.

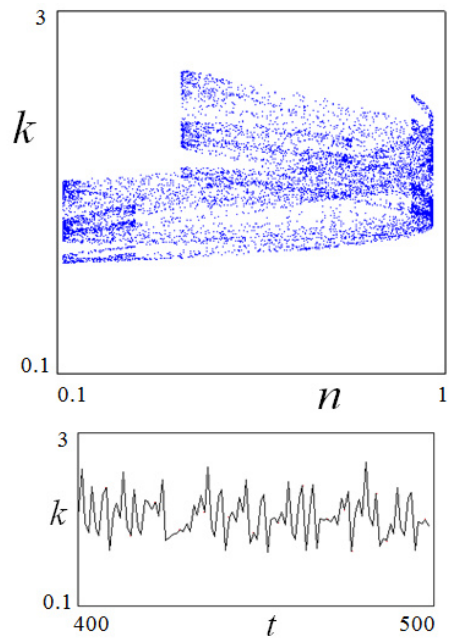

Figure 5: Chaotic attractor in the phase plane and corresponding dynamics of k. Parameters: $s_r = 0.6$, $a = 1.5$, $b = c = 2.9$, $\delta = 0.45$, $s_w = 0.15$ and $\mu = 3.88$.

5 Conclusions

We have considered a Kaldor-Pasinetti one-sector growth model with Leontief technology and logistic labor force growth rate. We have shown that these assumptions imply that the dynamics of the capital per worker is governed by a triangular, nonlinear and discontinuous two-dimensional dynamical system. Besides the typical convergence to a steady state, we have shown that complicated endogenous dynamics such as growth cycles and even chaotic dynamics characterize a large portion of the parameters' space. We have found new dynamic scenarios, not present in models where these assumptions are considered one at a time. The detailed investigation of the bifurcation mechanisms occurring in this model deserve of further studies.

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