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An Overview on D-stable Matrices

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An Overview on D-stable Matrices

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Abstract

We give an overview on the main properties of D-stable matrices, i.e. of those square matrices A for which the product DA is stable for *any* choice of the diagonal matrix D, with all positive diagonal elements. These matrices, introduced in economic analysis by Arrow and Mc Manus (1958), have found, besides applications in mathematical economics, applications also in other fields, such as mathematical ecology, population dynamics, the theory of electric circuits, analysis of neural networks, control theory and other questions.

Key words

D-stability of matrices, stability of matrices.

1. Introduction

D-stable matrices play an important role in various applications, especially in economic analysis (see, e.g., Arrow (1973), Arrow and McManus (1958), Giorgi (2003), Hahn (1982), Kemp and Kimura (1978), Newman (1959), Quirk and Saposnik (1968), Wods (1978)) but also in other fields, such as iterative numerical methods, studies on neural networks, circuits, large scale systems, mathematical ecology, etc.

A good reference work for these last applications is the book of Kaszkurewicz and Bhaya (2000).

A (real) square matrix A, of order n, is said to be *D-stable* if the matrix DA is (negative) stable (i.e. the real part of each its eigenvalue is negative) for *all* diagonal matrices D, with positive entries on their main diagonal: $d_{ii} > 0, \forall i = 1, 2, ..., n$.

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Obviously, if A is D-stable, then it is also stable (i.e. $Re(\lambda_i(A)) < 0$, for each i), but the converse does not necessarily hold. The aim of this paper is to give an overview on the problems concerning D-stability of real (square) matrices, with some new remarks.

The paper is organized as follows.

Section 2 is concerned with a short account on the economic motivations which gave rise to the studies on D-stable matrices.

Section 3 is concerned with the mathematical concepts and properties related to D-stability.

Section 4 investigates some necessary conditions for D-stability and some sufficient conditions.

Section 5 is concerned with some developments on the necessary and sufficient conditions for the D-stability of matrices of second and third order; some comments concerning the characterization of D-stable matrices the fourth order are made.

Section 6 contains some concluding remarks.

As already mentioned, all matrices considered are real. Moreover, we adopt the following notations, conventions and definitions.

- $\boldsymbol{N}$ denotes the set $\{1, 2, ..., n\}$;
- the determinant of A is denoted by $|A|$ or by $\det A$;
- $\mathfrak{D}$ is the class of real diagonal matrices D, i.e. $D = [d_{ij}]$, $i, j \in \boldsymbol{N}$, $i \neq j \Rightarrow d_{ij} = 0$;
- $\mathfrak{D}_+$ is the subclass of $\mathfrak{D}$ such that it holds $\mathrm{d}_{ii} > 0, \forall i \in \boldsymbol{N}$;
- $x = [0]$ is the zero vector of $\mathbb{R}^n$;
- the vector x is *nonnegative* ($x \geqq [0]$) if $x_i \geqq 0$, $\forall i \in \boldsymbol{N}$;
- the vector x is *semipositive* ($x \geq [0]$) if $x \geqq [0]$, but $x \neq [0]$;
- the vector x is *positive* ($x > [0]$) if $x_i > 0$, $\forall i \in \boldsymbol{N}$;
- the same conventions hold for matrices;
- the notations $x \leqq [0]$, $x \leq [0]$, $x < [0]$ are of obvious meaning;
- the square real matrix A is (negative) *stable* if all its eigenvalues have a negative real part;
- the matrix A is *Metzlerian* if $i \neq j \Rightarrow a_{ij} \geqq 0$;
- the square real matrix A is *negative quasi-definite* if $x \neq [0] \Rightarrow x^T Ax < 0$;
- the matrix A is *negative definite* if it is symmetric and negative quasi-definite;

- the matrix A has a *negative diagonal* if $a_{ii} < 0$, $\forall i \in \boldsymbol{N}$;
- the matrix A has a *quasi-dominant diagonal* (in the sense of McKenzie (1960)) if there exists a matrix $D \in \mathfrak{D}_+$ such that, with $B = AD$ or equivalently for $B = A^T D$, it holds

$$|b_{ii}| > \sum_{\substack{j \in \boldsymbol{N} \\ j \neq i}} |b_{ij}| \,, \forall i \in \boldsymbol{N}.$$

  If moreover, $a_{ii} < 0$, for all $i \in \boldsymbol{N}$, the A has a *negative quasi-dominant* diagonal.

- The *principal minors* of order k of A (square) are all the determinants made with k rows and the *corresponding* k columns of A. The sum t_k of all the $\binom{n}{k}$ principal minors of order k is called the *trace of order* k of A. The *North-West principal minors* or *leading principal minors* are the determinants made with the first k rows and first k columns, with $k = 1, 2, \ldots, n$.
- The square matrix A is *D-stable* if DA is stable for any $D \in \mathfrak{D}_+$ that is

$$D \in \mathfrak{D}_+ \;\Rightarrow\; DA \text{ is stable.}$$

  This is the classical definition of D-stable matrices, however, in economic analysis we encounter also a stronger definition, given in the pioneering paper of Arrow and McManus (1958) and that we call "strong D-stability": A is *strongly* D-stable if it holds

$$D \in \mathfrak{D} \;\Rightarrow\; \{DA \text{ is stable} \;\Leftrightarrow\; D \in \mathfrak{D}_+\}.$$

- The square matrix A is *Hicksian* if all its principal minors of order k have the sign of $(-1)^k$. A is *almost Hicksian* if, for each $i \in \boldsymbol{N}$, its principal minors of order i have the same sign of $(-1)^i$ or are zero, and their sum has the sign of $(-1)^i$. See, e.g., Quirk and Saposnik (1968). In the mathematical literature Hicksian matrices are also called "(NP)-matrices" and almost Hicksian matrices are also called "$(NP)_0^+$-matrices" (see Woods (1978), Johnson (1974), Kemp and Kimura (1978)).
- The square matrix A, where $a_{ij} \leqq 0, \forall i \neq j$, is a *K-matrix* (also called an "*M-matrix*") if it satisfies any one of the numerous equivalent conditions which characterize this class: see, e.g., Fiedler and Pták (1962), Poole and Boullion (1974), Plemmons (1977), Magnani and Meriggi (1981).

For example, given A (square) with $a_{ij} \leqq 0, \forall i \neq j$, the following conditions (which characterize the class of K-matrices) are mutually equivalent:

a) there exists an $x \geqq [0]$ such that $Ax > [0]$;
b) for any $c \geqq [0]$, there exists an $x \geqq [0]$ such that $Ax = c$;
c) the matrix A is nonsingular and $A^{-1} \leq [0]$;

d) all the principal minors of A are positive;
e) all the North-West principal minors of A are positive;
f) the real parts of all eigenvalues of A are positive (i.e. A is *positive stable*).

2. A short account on the main economic applications of D-stable matrices

In economic analysis D-stable matrices were first considered by Arrow and McManus (1958) in continuation of some questions posed in Enthoven and Arrow (1956); perhaps the pioneering paper of Arrow and McManus is indeed the first study concerned with D-stability of matrices. The interest of economists in this kind of problems arises from the study of the dynamic stability of the so-called "tâtonnement process" in a Walrasian model of general equilibrium.

Let us consider a model consisting of a set of n functional relations of the form $F_i(p)$, $i = 1, 2, ..., n$, where p is an n-dimensional real vector

$$p = [p_1, p_2, ..., p_n]$$

whose elements are the market prices of the n goods exchanged in a competitive market and where $F_i(p)$ is the "excess demand function" for good i at the price vector p, that is $F_i(p)$ is demand minus supply, given the price vector p. Usually p is taken to be a positive vector: $p > [0]$.

An *equilibrium* of the model is said to occur at an *equilibrium price vector* $\overline{p}$ if

$$F_i(\overline{p}) = 0, \forall i = 1, 2, ..., n.$$

Indeed, in this Walrasian competitive economy, an equilibrium occurs when all markets are simultaneously cleared, that is excess demand is zero in all markets.

In order to describe the dynamic behavior of the model, the classical approach of Samuelson (1944, 1947) takes into consideration an adjustment process (the so-called "tâtonnement process", described in an informal way by L. Walras) represented by the following system of first-order differential equations

$$p_i'(t) = g_i \left[F_i \left(p\left(t\right)\right)\right], \ i = 1, 2, ..., n,$$

where g_i is an increasing function of F_i, that is $dg_i/dF_i > 0, \forall i = 1, 2, ..., n$, and it is further assumed that $g_i(0) = 0, \ i = 1, 2, ..., n$.

We are interested in a linearized version of the above adjustment process (assuming the differentiability of each g_i and F_i) :

$$p_i' = g_i' \sum_{j=1}^{n} \frac{\partial F_i}{\partial p_j} \left(p_j - \overline{p}_j\right), \ i = 1, 2, ..., n,$$

where the linear approximation is taken with respect to an equilibrium price vector $\overline{p}$. We can write this in matrix terms as

$$p' = B\,(p - \overline{p})$$

where $B = DA$, D is diagonal matrix with diagonal entries $dg_i/\partial F_i$, and

$$A = \left[\frac{\partial F_i}{\partial p_j}\right], \; i, j = 1, 2, ..., n.$$

The elements of the above diagonal matrix D describe, respectively, the various "adjustment speeds" of the prices related to the markets. The more d_i is large, the more will be the variation of the i-*th* price, with respect to time, for a given value of the excess demand for the i-*th* good.

If the n speeds of the adjustment, described by D, are all equal, then D is a "scalar matrix" and the stability of $B = DA$ coincides with the stability of A, otherwise the stability of DA and the stability of A are distinct properties. Obviously, if A is D-stable, i.e. DA is stable for each $D \in \mathfrak{D}_+$, then A is stable (it is sufficient to choose $D = I$), but the converse is not true:

$$A \text{ is stable } \nRightarrow \; A \text{ is } D\text{-stable.}$$

It is sufficient to consider the following example, taken from Arrow an McManus (1958),

$$A = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

is stable, however

$$DA = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

is unstable.

We recall a remarkable result of Fisher and Fuller (1958); see also Fisher (1972):

- If A is a real Hicksian matrix, then there exists a diagonal matrix $D \in \mathfrak{D}_+$ such that the eigenvalues of DA are all real, negative and distinct.

From the above short notes, it appears unmistakable the importance of the role of D-stable matrices in the dynamic analysis of economic models.

Indeed, it is supposed that the various "speeds of adjustment" are not known *a priori* and therefore "there is interest in the class of cases in which stability of an equilibrium can be established independent of speeds of adjustment, that is, cases in which stability can be proved for all (positive) speeds of adjustment of markets" (Quirk (1981), page 125).

Likewise it is evident the importance to study the relations between D-stable matrices and other concepts and classes of stable matrices.

3. D-stable and related matrices

We recall once more that the square matrix A is *D-stable* if

$$D \in \mathfrak{D}_+ \ \Rightarrow \ DA \text{ is stable.} \tag{1}$$

Arrow and McManus (1958) introduced the following definition, which perhaps is the first definition where the term "D-stable matrices" appears. We call a square matrix A *strongly D-stable* if

$$D \in \mathfrak{D} \ \Rightarrow \ \{DA \text{ is stable} \Leftrightarrow D \in \mathfrak{D}_+\}.$$

Another class of matrices related to D-stable and to strongly D-stable matrices is the class of the *diagonally stable matrices*, also known as the *Volterra-Lyapunov stable matrices* or also *Hurwitz diagonally stable matrices*:

There exists a diagonal matrix $D \in \mathfrak{D}_+$ such that $DA + A^T D$ is negative definite.

A square matrix A is said to be *totally D-stable* when every submatrix of A, whose determinant is a principal minor of A, is D-stable.

A square matrix A is *additively D-stable* if $A - D$ is stable for all *nonnegative* diagonal matrices D.

Now we recall the notion of *sign-stable matrix* or *qualitatively stable matrix*.

If A is a square n-matrix, we define the matrix $\mathsf{sign}A$, as the matrix whose generic element $\mathsf{sign}(a_{ij})$ is given by

$$\mathsf{sign}(a_{ij}) = \begin{cases} +1 & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0 \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

Now, let $Q_A = \{B : \mathsf{sign}B = \mathsf{sign}A\}$, i.e. Q_A is the set of all square n-matrices B with the same sign pattern of A.

A is said to be *sign-stable* or *qualitatively stable* if every matrix of Q_A is stable.

A square matrix A is said to be *potentially stable* if there exists a stable matrix $B \in Q_A$.

Arrow and McManus (1958) introduced also the concept of *S-stability*:

A is said to be *S-stable* if SA is stable, for all (symmetric) definite positive matrices S. We note that Arrow and McManus give a stronger definition, we may call "strong S-stability":

A is said to be *strongly S-stable* if, for all symmetric matrices S, SA is stable if and only if S is positive definite.

However (see Johnson (1974 b)) the two definitions for S-stable matrices yield the same class of matrices.

We start from some basic results of Arrow and McManus (1958), not always considered in the mathematical literature, and from some results of Magnani (1972), published (in Italian) as an "Institute Report" and therefore not easily available to the interested readers.

Theorem 1 (Arrow and McManus (1958))

If there exists a diagonal matrix E such that $E^{-1}AE$ is either a stable Metzlerian matrix or a negative quasi-definite matrix, then A is strongly D-stable.

In the same paper Arrow and McManus make the following conjecture:

Conjecture 1

A is strongly D-stable if and only if it can be represented as

$$A = EME^{-1},$$

with $E \in \mathfrak{D}$ and M is either a Metzlerian stable matrix or a negative quasi-definite matrix.

This conjecture is false: it is sufficient to choose $n = 2$, $a_{11}a_{22} = 0$, $a_{11} + a_{22} < 0$, $a_{12}a_{21} < 0$. It can be proved that a matrix having these properties is D-stable and also strongly D-stable, but does not verify the assumptions of Conjecture 1.

It is quite obvious that the strong D-stability implies the usual D-stability. Apart from the paper of Arrow and McManus, strong D-stability has received a scarce attention from the researchers. We report the following results, which are a direct consequence of Theorem 1.

Theorem 2

If A admits the representation $A = (S + M)$, with S skew-symmetric ($S^T = -S$) and M negative quasi-definite, then A is strongly D-stable.

Theorem 3

If A is a Metzlerian matrix and $-A$ is a K-matrix (also called an "M-matrix"), then A is strongly D-stable.

The following results are taken from Magnani (1972).

Theorem 4

I) There exist matrices A which are D-stable only in the usual sense.

II) If A is D-stable or also strongly D-stable, then A is almost Hichsian.

III) If A is D-stable, but not strongly D-stable, because there exists a matrix

$$D^* \in \mathfrak{D},\ D^* \notin \mathfrak{D}_+, \quad \text{with } D^*A \text{ stable}, \tag{2}$$

then it holds $d^*_{ii} \neq 0$, $\forall i \in \boldsymbol{N}$ and D^* contains an even number of negative elements.

IV) Either if $n = 2$ or if A admits the representation $A = EME^{-1}$, with $E \in \mathfrak{D}$ and M a Metzlerian matrix, then A is D-stable (in the usual sense) if and only if A is strongly D-stable.

In order to prove Theorem 4, we recall the well-known *Routh-Hurwiz criterion* for the stability of a square matrix (see, e.g., Gantmacher (1966)).

Theorem 5

Let M be a square n-matrix. Let us denote by $S_i(M)$ its *trace of order* i, that is the sum of all its $\binom{n}{i}$ principal minors of order i, and let

$$K_i = K_i(M) = \begin{cases} (-1)^i S_i(M), & \forall i \in \mathbf{N} \\ 0, & \forall i > \mathbf{N}. \end{cases} \tag{3}$$

Then M is stable if and only if

$$\begin{vmatrix} K_1 & K_3 & K_5 & \ldots & K_{2i-1} \\ 1 & K_2 & K_4 & \ldots & K_{2i-2} \\ 0 & K_1 & K_3 & \ldots & K_{2i-3} \\ 0 & 1 & K_2 & \ldots & K_{2i-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & K_i \end{vmatrix} > 0, \ \forall i \in \mathbf{N}. \tag{4}$$

Note that (4) implies

$$K_i > 0, \ \forall i \in \mathbf{N}.$$

Proof of Theorem 4

I) It is sufficient to consider the following pair (A, D^*)

$$\left\{ \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & -2 \end{bmatrix} ; \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

in which A verifies (1), but D^* verifies (2).

II) Setting $M = DA, \ D \in \mathfrak{D}_+$, from Theorem 5 we obtain at once the thesis of II). (See also Quirk and Ruppert (1965)).

III) Being $I \in \mathfrak{D}_+$, we get that $IA = A$ and D^*A are stable matrices; from Theorem 5 we obtain $(-1)^n |A| > 0$, $(-1)^n |D^*A| > 0$, i.e. $|D^*| = d^*_{11}, d^*_{22}, ..., d^*_{nn} > 0$, i.e. the thesis of III).

IV) We have already remarked that the strong D-stability implies the usual D-stability. Let us suppose that, with $n = 2$, A is D-stable, but not strongly D-stable, i.e. (1) holds but there exists a matrix D^* verifying (2). From II) and III) of the present theorem, we have, with $n = 2$,

$$a_{11} \leqq 0, a_{22} \leqq 0, \ a_{11} + a_{22} < 0, \ d^*_{11} < 0, \ d^*_{22} < 0$$

and therefore

$$d^*_{11} a_{11} + d^*_{22} a_{22} > 0,$$

whereas with $n = 2$ the stability of D^*A implies, by virtue of Theorem 5, the opposite inequality. Therefore, with $n = 2$ the usual D-stability implies the strong D-stability.

Let us now suppose that $A = EME^{-1}$ is D-stable, with $E \in \mathfrak{D}$ and M a Metzlerian matrix. Therefore $IA = A$ is stable, as well as $M = E^{-1}AE$, because M is given by a similarity transformation. Therefore M is a stable Metzlerian matrix and from Theorem 1 we get the strong D-stability of A. The thesis IV) is therefore proved. □

In the same paper Magnani (1972) discusses the following conjecture.

Conjecture 2

The matrix A is D-stable (in the usual sense or in the strong sense) if and only if it satisfies at least one of the following properties:

1) The matrix A admits the representation

$$A = EME^{-1} \text{ with } E \in \mathfrak{D} \text{ and } M \text{ a Metzlerian matrix.} \tag{5}$$

2) The matrix A admits the representation

$$A = EME^{-1} \text{ with } E \in \mathfrak{D} \text{ and } M \text{ a negative quasi-definite matrix.} \tag{6}$$

3) There exists a matrix $C \in \mathfrak{D}_+$ such that $(CA + A^T C)$ is negative definite.

4) The matrix A has a negative quasi-dominant diagonal.

5) The matrix A is qualitatively stable.

Before discussing the validity of Conjecture 2 we recall the following basic results.

Theorem 6

If there exists a matrix $C \in \mathfrak{D}_+$ such that $(CA + A^T C)$ is negative definite, then A is D-stable.

In other words, if A is diagonally stable, then A is D-stable. This classical result is either attributed to Arrow and McManus (1958) or is obtained from the well-known Lyapunov criterion for the stability of a matrix: the square matrix A is stable if and only if there exists a symmetric positive definite matrix B such that $\left(BA + A^T B\right)$ is negative definite. See, e.g., Gantmacher (1966).

Theorem 7

If the matrix A has a negative quasi-dominant diagonal, then A is D-stable.

This result is due to Mckenzie (1960); see also Quirk and Saposnik (1968) and Beavis and Dobbs (1990).

Theorem 8

If the matrix A is qualitatively stable, then A is D-stable.

This is a result due to Quirk and Ruppert (1965). On the grounds of the above results Magnani (1972) discusses Conjecture 2 by means of the following theorem.

Theorem 9

Let A be a square matrix of order n, with

$$n = 2,\ a_{11} < 0,\ a_{22} < 0 \tag{7}$$

Then, Conjecture 2 holds and every matrix D-stable (or strongly D-stable) for which (7) is verified, satisfies at least 3 properties among the 5 properties listed in Conjecture 2. If (7) is not verified, Conjecture 2 is in general false.

Proof

First of all, we remark that, thanks to Theorem 1, properties 1) and 2) of Conjecture 2 imply the strong D-stability of A, whereas the usual D-stability follows from properties 3), 4) and 5) of the Conjecture 2, by virtue of Theorems 6, 7 and 8.

Let us begin to check if, under assumption (7), the D-stability (usual or strong) of A implies someone of properties 1) - 5) of Conjecture 2.

Let us consider the matrix

$$\begin{aligned} &E = [e_{ij}]\,,\ i,j = \{1,2\}\,,\ \text{with } e_{12} = e_{21} = 0,\ e_{11} = 1 \text{ and with} \\ &e_{22} = 1, \\ &e_{22} = \left| a_{21} / \left(a_{11}a_{22}\right)^{1/2} \right|, \\ &e_{22} = \left| \left(a_{11}a_{22}\right)^{1/2} / a_{12} \right|, \\ &e_{22} = \left| \left(a_{21}/a_{12}\right)^{1/2} \right|, \end{aligned} \tag{8}$$

depending on whether in (7) it holds, respectively

$$\begin{aligned} &a_{12} = a_{21} = 0, \\ &a_{12} = 0 \neq a_{21}, \\ &a_{12} \neq 0 = a_{21}, \\ &a_{12} \neq 0 \neq a_{21}. \end{aligned} \tag{9}$$

Obviously, $E \in \mathfrak{D}_+$ and the matrix

$$M = [m_{ij}] = E^{-1}AE = [a_{ij}e_{jj}/e_{ii}] \tag{10}$$

verifies the following properties

$$\begin{aligned} &m_{11} = a_{11}; \\ &\left|M + M^T\right| = 4a_{11}a_{22} - \left\{a_{12}e_{22} + (a_{21}/e_{22})\right\}^2 = \\ &\qquad = \begin{cases} 4a_{11}a_{22}, & \text{if either } \ a_{12} = a_{21} = 0 \text{ or } a_{12}a_{21} < 0; \\ 3a_{11}a_{22}, & \text{if either } \ a_{12} = 0 \neq a_{21} \text{ or } a_{12} \neq 0 = a_{21}; \\ 4\left|A\right|, & \text{if } \ a_{12}a_{21} > 0. \end{cases} \end{aligned}$$

Therefore, if A is D-stable, from (7) and from Theorem 5, with $n = 2$, we have $m_{11} < 0$, $\left| M + M^T \right| > 0$, i.e. $(M + M^T)$ is negative definite, that is M is quasi-negative definite. As (10) is equivalent to (6) and $E \in \mathfrak{D}_+$, it is evident that if (7) holds and A is D-stable, then A satisfies property 2) of Conjecture 2.

We note, moreover, that, under the same assumptions, property 3) of Conjecture 2 is equivalent to the existence of a pair (c_{11}, c_{22}) which solves the system

$$\begin{cases} c_{11} > 0, \quad c_{22} > 0, \\ 4a_{11}a_{22} - \left\{ \left| (c_{11}/c_{22})^{1/2} \right| a_{12} + \left| (c_{22}/c_{11})^{1/2} \right| a_{21} \right\}^2 > 0, \end{cases}$$

system which surely admits the solution $c_{11} = (e_{22})^2$, $c_{22} = 1$, with e_{22} chosen as in (8) - (9). Therefore, under assumption (7), if the matrix A is D-stable, it satisfies property 3) of Conjecture 2.

Now let us suppose that A is D-stable, that (7) holds and that the following assumptions hold:

$$\text{either} \quad (a_{12} \geqq 0,\ a_{21} \geqq 0) \quad \text{or} \quad (a_{12} \leqq 0,\ a_{21} \leqq 0). \tag{11}$$

In the first case A is a Metzlerian matrix and it is possible to chose $E = I$ in order to obtain $M = E^{-1}AE = A$, a Metzlerian matrix. In the second case we can choose

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in order to obtain

$$M = E^{-1}AE = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{12} & a_{22} \end{bmatrix},$$

again a Metzlerian matrix. We note that in both cases the following properties hold:

i) $E \in \mathfrak{D}$;

ii) the D-stability of A implies the stability of $IA = A$, because $I \in \mathfrak{D}_+$;

iii) in relations (3) it holds $K_i(M) = K_i(A) > 0$, $i = 1, 2$; therefore the matrices $A = EME^{-1}$ and $M = E^{-1}AE$ are stable;

iv) it holds $m_{ii} = a_{ii}$, $|m_{ij}| = |a_{ij}|$, $i, j \in \{1, 2\}$.

A classical result of McKenzie (1960) (generalized in the next Theorem 11) states that a Metzlerian matrix, is stable if and only if it has a negative quasi-dominant diagonal. It is therefore evident that in both cases (11) A admits the representation described in relation (5), with A stable Metzlerian matrix with a negative quasi-dominant diagonal. So A verifies properties 1 and 4 of Conjecture 2.

Let us now suppose that, under assumption (7), the matrix A is D-stable, but with $a_{12}a_{21} < 0$. In this evenience, for any matrix $M = E^{-1}AE$, defined by relation (5) with $E \in \mathfrak{D}$, it holds $m_{12}m_{21} = a_{12}a_{21} < 0$, therefore M cannot be a Metzlerian matrix and A cannot verify property 1) of Conjecture 2. The matrix A will satisfy property 4) if and only if $(a_{11}a_{22} + a_{12}a_{21}) > 0$: this inequality is not, however, implied by the D-stability of A, nor by relation (7).

As for what concerns property 5), we note that, according to Theorems 4 (item IV) and 5, under assumption (7), the same property is equivalent to $a_{12}a_{21} \leqq 0$.

It is therefore evident that, under assumptions (7), if A is D-stable (that is if $a_{11}a_{22} - a_{12}a_{21} > 0$), with $(a_{12}a_{21} > 0)$, $(a_{12}a_{21} < 0 < a_{11}a_{22} + a_{12}a_{21})$, $(a_{12}a_{21} < 0 \not< a_{11}a_{22} + a_{12}a_{21})$, $(a_{12}a_{21} = 0)$, A satisfies, respectively, properties $(1,2,3,4)$, $(2,3,4,5)$, $(2,3,5)$, $(1,2,3,4,5)$ of Conjecture 2.

Magnani (1972) then proves that the following matrix

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix},$$

which does not satisfy assumtions (7), is D-stable and also strongly D-stable, even if it does not verify any one of properties 1) - 5) of Conjecture 2. □

We note that the class of D-stable matrices is closed with respect to the multiplication by a positive scalar, but, as remarked by Johnson (1975), it is not closed with respect to addition. The example taken into consideration by Johnson is the following one:

$$A = \begin{bmatrix} -5 & 2 \\ 12 & -5 \end{bmatrix}; \qquad B = A^T.$$

It may be seen (see, e.g., Johnson (1974) and Section 5) that both A and B are D-stable, but their sum

$$A + B = \begin{bmatrix} -10 & 14 \\ 14 & -10 \end{bmatrix}$$

is not D-stable, because $\det(A + B) < 0$ and this matrix, therefore, must have one negative and one positive eigenvalue.

We now describe some relationships between various classes of matrices and the class of D-stable matrices (see, e.g., Kemp and Kimura (1978), Giorgi (2003), Quink (1981), Hershkwitz (1992), Newman (1959)). Some results have already been given and other results will be given in the next Section.

Theorem 10

Let A be a real square matrix. Then the following relationships hold.

i) A is negative quasi-definite $\Leftrightarrow$ A^T is negative quasi-definite $\Leftrightarrow$ $\Leftrightarrow$ A^{-1} is negative quasi-definite $\Rightarrow$ A is S-stable $\Rightarrow$ A is D-stable $\Rightarrow$ $\Rightarrow$ A is stable $\Leftrightarrow$ BAB^{-1} is stable for some (non-singular) matrix B.

ii) A is negative quasi-definite $\Rightarrow$ A is totally stable $\Rightarrow$ A is D-stable and $A \in (NP)$ (that is, A is Hicksian).

iii) A has a negative quasi-dominant diagonal $\Rightarrow$ A is totally stable.

iv) A is qualitatively stable $\Rightarrow$ A is D-stable $\Rightarrow$ A is stable.

v) A is qualitatively stable with all diagonal elements $a_{ii} < 0 \Rightarrow A$ is totally stable.

vi) A has all diagonal elements a_{ii} negative $\Rightarrow$ A is a potentially stable matrix.

Theorem11

Let A be a Metzlerian matrix $(a_{ij} \geqq 0, \ \forall i \neq j)$. Then the following equivalences hold:

$$A \text{ is stable} \Leftrightarrow A \text{ is } D\text{-stable} \Leftrightarrow A \text{ is totally stable} \Leftrightarrow$$
$$\Leftrightarrow A \text{ is Hicksian} \Leftrightarrow A \text{ has a negative quasi-dominant diagonal.}$$

We note that if A is a Metzlerian matrix, then $-A \in Z$, Z being the class of matrices with nonpositive off-diagonal elements, in the terminology of Fiedler and Pták (1962). Therefore the equivalences of Theorem 11 can be reformulated so that a matrix $A \in Z$ belongs to the class of K-matrices (or M-matrices).

We recall also (Theorem 3) that, under the Metzlerian assumption, we can assert that A is stable if and only if A is *strongly* D-stable (i.e. strong D-stability and D-stability coincide).

Theorem 11 holds also for a larger class than Metzlerian matrices, the so-called *Morishima matrices* (Morishima (1952, 1970)).

Definition 1

A square matrix A of order n is a *Morishima matrix* if there exists a permutation matrix Π such that

$$\Pi A \Pi^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{12}$$

where A_{11} and A_{22} are square and Metzlerian and $A_{12} \leqq [0]$, $A_{21} \leqq [0]$.

Morshima (1952) requires the stronger condition that A_{11} and A_{22} are to be square and non negative. Moreover, he proves that if $a_{ij} \neq 0$, $\forall i, j = 1, 2, ..., n$, then this stronger requirement is equivalent to $a_{ii} > 0$, $\forall i$; $\mathsf{sign} a_{ij} = \mathsf{sign} a_{ji}$, $\forall i \neq j$; $\mathsf{sign} a_{ij} = \mathsf{sign} a_{ik} a_{kj}$ for any i, j, k distinct.

Theorem 12

Theorem 11 also holds under the assumption that A is a Morishima matrix.

Proof

See Kemp and Kimura (1978).

At the end of the present section we wish to stress that the already recalled result of Fisher and Fuller (1958) is of another nature, with respect to the problem of D-stability: this result, indeed, establishes that if A is an Hicksian matrix, then there is assured the existence of a diagonal matrix $D^* \in \mathfrak{D}_+$, such that D^*A is stable, with all negative and simple eigenvalues.

Besides the simpler proof of Fisher (1972), one may see the proof of Ballantine (1970), who generalizes the above result also to complex matrices. The question has been recently reconsidered by Locatelli and Schiavoni (2012), who give a necessary and sufficient condition for the existence of a diagonal matrix D^* such that D^*A is stable, together with all its principal submatrix. They also give an explicit formula to find the diagonal matrix D^*.

4. Necessary conditions for D-stability. Sufficient conditions for D-stability

Perhaps one of the most old necessary condition for the D-stability of a real matrix is a slight variant of a condition due to Metzler (1945), which, however, concerns totally D-stable matrices, and not D-stable matrices, as postulated by Metzler.

Theorem 13

Let A be a real matrix of order n; if A is totally D-stable, then $A \in (NP)$, i.e. A is *Hicksian*: all its principal minors of order i $(i = 1, 2, \ldots, n)$ have the sign of $(-1)^i$.

The Metzler condition has subsequently been taken into consideration by Arrow (1973) who established a necessary condition for A to be D-stable. See also Johnson (1974 a) and Quirk and Ruppert (1965).

Theorem 14

Let A be a real matrix of order n; if A is D-stable, then $A \in (NP)_0^+$, i.e. A is *almost Hicksian*: all its principal minors of order i $(i = 1, 2, \ldots, n)$ are nonnegative, if i is even, and nonpositive if i is odd and, moreover, there exists at least one principal minor of order i $(i = 1 = 1, 2, \ldots, n)$ different from zero.

The reverse implication does not hold, unless $n = 2$: in this case (see Section 5) the conditions of Theorem 14 are both necessary and sufficient for A to be D-stable.

Another necessary condition for D-stability, equivalent to the one of Theorem 14, has been given by Johnson (1974 a); however, this last condition cannot be performed in a finite number of calculations. A flaw in the formulation of Johnson has been subsequently corrected by Cross (1978).

We remark that if A is D-stable, then A is non-singular and each of the following matrices is D-stable:

a) A^{-1};

b) P^TAP, where P is any permutation matrix;

c) DAE, with $D, E \in \mathfrak{D}_+$;

d) A^T.

A classical paper in which several sufficient conditions for D-stability are presented is the paper by Johnson (1974 b). Some conditions have already been discussed in the previous Section.

Each of the following conditions is sufficient for A to be D-stable.

I) There exists a diagonal matrix $D \in \mathfrak{D}_+$ such that $(DA + A^T D)$ is negative definite. See Theorem 6. This is essentially the condition given by Arrow and McManus (1958). In other words, if A is diagonally stable (Volterra-Lyapunov stable), then A is D-stable. That the present criterion is not necessary is shown by Johnson (1974 c). This important criterion has, however, a weak relevance from a practical (i.e. computational) point of view. Khalil (1980) presented an algorithm in which the existence of the diagonal matrix $D \in \mathfrak{D}_+$ is verified by means of a convex minimization problem. Define the function $g : \mathbb{R}^n \to \mathbb{R}$ as

$$g(p) = \lambda_{\max}(DA + A^T D),$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of the respective matrix, and define $V = \{p \in \mathbb{R}^n : 0 \leqq p_i \leqq 1\}$. It can be verified that V is a convex compact set and that $g(p)$ is a continuous convex function. Khalil (1980) proves the following result.

Theorem 15

There exists a positive diagonal matrix D such that $(DA + A^T D)$ is negative definite if and only if

$$\min_{p \in V} g(p) < 0.$$

Efficient numerical algorithms for such problems are discussed in the specialized literature of mathematical programming.

II) The matrix A has a negative quasi-dominant diagonal (see Theorems 7 and 10). This is the condition due to McKenzie (1960). This condition is finitely verifiable by means of the theory of K-matrices (or M-matrices). A matrix A satisfies the above quasi-dominance condition if and only if the matrix $C = [c_{ij}]$ defined by

$$\begin{aligned} c_{ij} &= -|a_{ij}|, \ i \neq j, \\ c_{ii} &= -a_{ii} \end{aligned}$$

is a K-matrix.

III) The matrix $-A$ is a K-matrix; in other words (see Theorem 3) A is Metzlerian and $-A$ verifies any one of the equivalent characterizations of the K-matrices (or M-matrices) given by Fiedler and Pták (1962) and by the other authors quoted in the Introduction.

IV) A quite trivial condition for D-stability is that A is triangular, with $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$. An even more trivial condition is that A is diagonal with all negative diagonal elements.

V) The matrix A is *qualitatively stable* or *sign-stable*. The sign-stable matrices have been introduced by Quick and Ruppert (1965). We have already listed in Theorem 10 the present sufficient condition for D-stability. As Johnson (1974 b) remarks, "sign stability is an essential combinatorial quality" and the most obvious examples of sign-stable matrices are the real matrices satisfying conditions IV) above. Sign-stable matrices have been characterized by Quirk and Ruppert (1965) for the case where A has all diagonal elements negative, and by Jeffries, Klee and Van Den Driessche (1977) for the general case. We give only the theorem of Quirk and Ruppert; first we need the notion of "cycle" of a square matrix A of order n (see e.g., Maybee and Quirk (1969)).

For the nxn matrix A, we call the product

$$A_p = a_{i(1)i(2)}a_{i(2)i(3)}...a_{i(r-1)i(r)}$$

a *path* or *chain* in A if $r \geqq 2$ and the set

$$V_p = \{i(1),\ i(2), ...,\ i(r-1)\}$$

consists of distinct elements of $\boldsymbol{N}$. The number $r-1$ is called the *length* of A_p and the set V_p is called the *index* set of A_p.

The product

$$A_c = a_{i(1)i(2)}a_{i(2)}a_{i(3)}...a_{i(r)i(1)}$$

is called a *cycle* of A if $r \geqq 2$ and the product

$$a_{i(1)i(2)}a_{i(2)i(3)}...a_{i(r-1)i(r)}$$

is a path in A.

The *length of the cycle* is the number r, and the set $V_c = \{i(1),\ i(2), ...,\ i(r)\}$ is called the *index set* of the cycle.

The diagonal entries $a_{i(q)i(q)}$ can also be considered cycles of A and they have length one and index set consisting of the single index $V_c = \{i(q)\}$.

Theorem 16 (Quirk and Ruppert (1965))

Let A be a square matrix of order n with all diagonal elements negative. Then A is sign-stable if and only if

i) every cycle in A of length 2 is non-positive;

ii) every cycle in A of length 3 or more is zero.

VI) The matrix A is *tridiagonal* (or a *Jacobi matrix*) and A is Hicksian. We recall that $A = [a_{ij}]$ is tridiagonal if $a_{ij} = 0$ whenever $|i-j| > 1$.

VII) The matrix A is *oscillatory*, that is, every i^{th} order minor of A has $\mathsf{sign}(-1)^i$ or zero and for some power of A every i^{th}-order minor has the sign of $(-1)^i$.

VIII) For each $x \neq [0]$ there exists $D \in \mathfrak{D}$ such that $x^T DAx < 0$.

IX) The Hadamard product of P and A, $P \circ A \equiv (p_{ij}a_{ij})$, is stable for every positive definite symmetric matrix P.

X) The matrix A is Hicksian stable and A is (strongly) sign-symmetric; that is, every pair of symmetrically placed minors of A has a nonnegative product.

Carlson and Johnson (1974) formulated the conjecture that A is D-stable if and only if $(A - D)$ is stable for each nonnegative diagonal matrix D. We have called "additively D-stable matrices" the matrices satisfying the above property. Subsequently, G. W. Cross (1978) has shown that in general D-stability does not imply additive D-stability, and that additive D-stability does not imply D-stability.

However, the same author proved that for matrices of order 2 and 3 additive D-stability is both necessary and sufficient for D-stability. Moreover, if A is Volterra-Lyapunov stable, then it is additively D-stable (Cross (1978) Proposition 1). Another interesting result of this author is contained in the following theorem (we recall that an n-square matrix A (also complex) is called *normal* if $A\overline{A}^T = \overline{A}^T A$, where $\overline{A}$ is the conjugate of A. Normal matrices include diagonal, real symmetric, real skew-symmetric, orthogonal, Hermitian, skew-Hermitian and unitary matrices.

Theorem 17 (see also Berman and Hershkowitz (1983))

For normal matrices and for Metzlerian matrices, additive stability, D-stability and Volterra-Lyapunov stability are equivalent.

For the reader's convenience we report two diagrams, taken from Hershkowitz (1992), showing the implication relations between some of the classes of matrices previously considered.

A is Volterra-Lyapunov stable

↙ ↓ ↘

A is Hicksian (i.e. $A \in NP$) | A is D-stable | A is additively D-stable

↓ ↘ ↙

A is stable | A is almost Hicksian (i.e. $A \in (NP)_0^+$)

Let A be Metzlerian; then the following diagram holds.

	A is Volterra-Lyapunov stable	
↙↗	↓↑	↖↘
A is Hicksian	A is D-stable	A is additively D-stable
↓↑	↓↑	↗↙
A is a nonsingular K-matrix (or M-matrix)	A is stable	

There are other sufficient conditions for D-stability: see, e.g., Datta (1978) and Kimura (1981). However, also these conditions, in the same way as those previously described, are in general not finitely verifiable.

Another important notion is the concept of "robust D-stability", introduced by E. H. Abed (1986), under the name "strong stability", and studied extensively by Kafri (2002). A square matrix A of order n is *robustly D-stable* if there is a scalar $\alpha > 0$ such that $(A + G)$ is D-stable for each square matrix G of order n, having a norm less than α.

Kafri (2002) extends to robust D-stability the sufficient conditions for D-stability given by Johnson (1974 b).

5. The problem of the characterization of D-stable matrices

First we report five necessary and sufficient conditions for the D-stability of a square matrix. The first two are trivial extensions of well-known criteria for the stability of a square matrix and no one of these five conditions can be considered as a true test for D-stability, as none of the said conditions is finitely verifiable. Under this aspect, following the words of J. P. Quirk (see Greenberg and Maybee (1981), especially the Discussion on Session II, pages 195-196), the search for a test of D-stability of practical use, for a square matrix of order n, seems to be hopelessly complicated. These efforts have been successful only for matrices of very low order ($n = 2, 3, 4$).

Necessary and sufficient conditions for A to be D-stable are:

I) The matrix DA satisfies the test of Routh-Hurwitz (Theorem 5) for any diagonal matrix $D \in \mathfrak{D}_+$.

II) The matrix DA verifies the condition of Lyapunov for any $D \in \mathfrak{D}_+$: there exists a symmetric positive definite matrix B such that

$$BDA + A^T DB$$

is negative definite for any $D \in \mathfrak{D}_+$.

III) See Johnson (1974 b, 1975). Suppose that DA is stable for a diagonal matrix $D \in \mathfrak{D}_+$. Then A is D-stable if and only if $A \pm iD$ is non-singular for all $D \in \mathfrak{D}_+$.

IV) See R. Johnson and A. Tesi (1999), Proposition 2.3: necessary and sufficient conditions for D-stability of A are stability of A and

$$\det\begin{pmatrix} A & D \\ -D & A \end{pmatrix} \neq 0, \text{ for all } D \in \mathfrak{D}_+ \,.$$

We remark that, being A non-singular, the above inequality can be equivalently written as

$$\det\left(A + DA^{-1}D\right) \neq 0, \text{ for all } D \in \mathfrak{D}_+.$$

V) See Johnson (1975). We need a formal notation for principal minors of a square matrix A of order n. Let $A_s = A_{i_1,\ i_2,\dots,\ i_k}$ denote the determinant of the principal submatrix determined by deleting from A the rows and columns indicated by the index set

$$S = \{i_1,\ i_2, \dots,\ i_k\} \subseteq \boldsymbol{N} = \{1, 2, \dots, n\}\,.$$

When $k = 0, S = \emptyset$ occurs, and when $k = n,\ S = \boldsymbol{N}$ occurs. By convention, $A_\emptyset = \det A$ and $A_{\boldsymbol{N}} = 1$. Moreover, let us define the following polynomials, where $X \in \mathfrak{D}$:

$$P_A(x_1,\ x_2, \dots,\ x_n) = R_e\left(\det[A + iX]\right) =$$
$$= \sum_{\substack{0 \leqq k \leqq n \\ k \text{ even}}} \left((-1)^{k/2} \sum_{\{i_1,\ i_2,\dots,\ i_k\} \subseteq \boldsymbol{N}} A_{i_1,\ i_2,\dots,\ i_k} x_{i_1}, x_{i_2}, \dots, x_{i_k} \right)$$

and

$$Q_A(x_1, x_2, \dots, x_n) = I_m\left(\det[A + iX]\right) =$$
$$= \sum_{\substack{1 \leqq k \leqq n \\ k \text{ odd}}} \left((-1)^{(k-1)/2} \sum_{\{i_1,\ i_2,\dots,\ i_k\} \subseteq \boldsymbol{N}} A_{i_1,\ i_2,\dots,\ i_k} x_{i_1}, x_{i_2}, \dots, x_{i_k} \right)$$

Then P_A and Q_A are polynomial in $x_1, x_2, \dots, x_n$, which are the elements of the diagonal matrix X.

Johnson (1975) proves the following result.

Theorem 18

Suppose that A (real and square) is stable. Then A is D-stable if and only if the system

$$\begin{cases} P_A(x_1,\ x_2, \dots,\ x_n) = 0 \\ Q_A(x_1,\ x_2, \dots,\ x_n) = 0 \end{cases}$$

has no solution $x_1 > 0, x_2 > 0, \dots, x_n > 0$.

An interesting consequence of the above theorem is that it seems possible to link the characterization of D-stable matrices to a polynomial programming problem; this idea has been exploited by Kanovei and Logofet (1998) to construct a test for the D-stability of matrices of order four.

Beyond the trivial cases of diagonal matrices and triangular matrices (see the point IV) of Section 4 and the case of Metzlerian matrices and also the case of Morishima matrices; see Theorem 12), the search for a finitely verifiable test for general square matrices is a very hard problem.

For $n = 2$ the problem is easy, but already for $n \geqq 3$ the problem becomes more and more complicate. The characterization of D-stable matrices of order 2 is due to Johnson (1974). We recall, following Fiedler and Pták (1966) and Johnson (1974), that a square matrix A of order n belongs to the class $(NP)_0^+$ if all principal minors of A of order i ($i = 1, 2, \ldots, n$) are nonpositive for i odd and nonnegative for i even, and at least one of the said minors of order i has the sign of $(-1)^i$. In the economic literature the class of matrices belonging to $(NP)_0^+$ is known also as the class of *almost Hicksian matrices* (see, e.g., Quirk and Ruppert (1965), Quirk (1981)).

Theorem 19 (Johnson (1974))

Let A be a square matrix of order $n = 2$; then A is D-stable if and only if $A \in (NP)_0^+$.

We recall also that $A \in (NP)_0^+$ is a necessary condition for the D-stability of a square matrix of any order (see Theorem 14).

The characterization for $n = 3$ was given, independently and with different proofs, by Cain (1976) and by Magnani (1990). For the reader's convenience we report the proof of Magnani, less short but more direct and elementary than the proof of Cain, and considering also the fact that the proof of Magnani has never been published.

We first remark that for $n = 3$, the condition $A \in (NP)_0^+$ simply means:

$$a_{11} \leqq 0,\ a_{22} \leqq 0,\ a_{33} \leqq 0,\ a_{11} + a_{22} + a_{33} < 0, \tag{13}$$

$$m_1 \geqq 0,\ m_2 \geqq 0,\ m_3 \geqq 0,\ m_1 + m_2 + m_3 > 0, \tag{14}$$

$$\det A < 0,$$

where m_i is the cofactor of a_{ii} .

Theorem 20

Let A be a real square matrix of order 3 and let m_i denote the cofactor of a_{ii} . Define

$$t = \det A + \left\{ (-m_1 a_{11})^{1/2} + (-m_2 a_{22})^{1/2} + (-m_3 a_{33})^{1/2} \right\}^2 . \tag{15}$$

Then A is D-stable if and only if $A \in (NP)_0^+$ and:

$$\text{either} \quad t > 0 \quad \text{if} \quad \{ m_i = 0 \ \Leftrightarrow \ a_{ii} = 0 \} \quad \text{holds;} \tag{16}$$

or

$$t \geqq 0 \ \text{ if } \ \{m_i = 0 \ \Leftrightarrow \ a_{ii} = 0\} \ \text{ does not hold.} \tag{17}$$

Proof

As the class of D-stable matrices is closed under multiplication by a positive scalar, we may normalize D by choosing

$$d_3 = 1,$$

so that $D \in \mathfrak{D}_+$ will be identified by the vector $d = [d_1, \ d_2] > [0]$. Now compute

$$\begin{aligned} k_1 &= -d_1 a_{11} - d_2 a_{22} - a_{33}, \\ k_2 &= d_2 m_1 + d_1 m_2 + d_1 d_2 m_3, \\ k_3 &= -d_1 d_2 \det A \end{aligned}$$

and apply the standard stability test of Routh-Hurwitz (Theorem 5), to DA: A is D-stable if and only if

$$\begin{cases} k_1 > 0, \ k_3 > 0, & \forall d > [0], \\ k_1 k_2 - k_3 > 0, & \forall d > [0]. \end{cases} \tag{18}$$

Assume $A \in (NP)_0^+$, as this condition is necessary for D-stability. This ensures

$$k_1 > 0, \ k_2 > 0, \ k_3 > 0, \quad \forall d > [0],$$

so we must check only that (18) holds if and only if (16) or (17) hold. Now, define

$$f = f(d) = k_1 k_2 - k_3,$$

$$\Delta = -m_1 a_{11} - m_2 a_{22} - m_3 a_{33} + \det A, \tag{19}$$

$$\alpha = \alpha(d_1) = -a_{22}(d_1 m_3 + m_1), \tag{20}$$

$$\beta = \beta(d_1) = -m_3 a_{11} (d_1)^2 + d_1 \Delta - m_1 a_{33}, \tag{21}$$

$$\gamma = \gamma(d_1) = d_1 m_2 (-d_1 a_{11} - a_{33}). \tag{22}$$

Then (18) is equivalent to

$$f(d) > 0, \ \forall d > [0] \tag{23}$$

and, being

$$f = \alpha (d_2)^2 + \beta (d_2) + \gamma,$$

(23) means

$$f / d_2 = \alpha d_2 + \beta + \gamma / d_2 > 0, \ \forall d > [0]. \tag{24}$$

Consider first the case where

$$m_2 a_{22} (m_1 + m_3)(a_{11} + a_{33}) > 0. \tag{25}$$

According to (13), (14), (20) and (22), the relations

$$\alpha > 0,\ \gamma > 0,\ \forall d > [0]$$

follow; hence, for each fixed $d_1 > 0$, f/d_2 attains its global minimum over $d_2 > 0$ when $d_2 = \delta_2 = (\gamma/\alpha)^{1/2}$. As

$$f(\delta_2)/\delta_2 = \beta + 2(\alpha\gamma)^{1/2},$$

(24) means

$$g = g(d_1) = \left\{\beta + 2(\alpha\gamma)^{1/2}\right\}/d_1 > 0,\ \forall d > [0], \tag{26}$$

that is

$$\begin{aligned} g \;=\; & \Delta + (-d_1 m_3 a_{11} - m_1 a_{33}/d_1) + \\ & +2\left\{-m_2 a_{22}\left[(-d_1 m_3 a_{11} - m_1 a_{33}/d_1) - m_1 a_{11} - m_3 a_{33}\right]\right\}^{1/2} > 0,\ \forall d_1 > [0]. \end{aligned} \tag{27}$$

It is useful to treat in a separate way the following four cases:

$$m_3 a_{11} < 0,\ m_1 a_{33} < 0, \tag{28}$$

$$m_3 a_{11} = 0,\ m_1 a_{33} < 0, \tag{29}$$

$$m_3 a_{11} < 0,\ m_1 a_{33} = 0, \tag{30}$$

$$m_3 a_{11} = 0,\ m_1 a_{33} = 0, \tag{31}$$

and to check that, by virtue of (13) and (14), (25) ensures that the paths of the signs the vectors

$$a = [a_{11},\ a_{22},\ a_{33}], \quad m = [m_1,\ m_2,\ m_3],$$

are only of the following kind:

- in the case (28): $[-\ -\ -]$; $[+\ +\ +]$;
- in the case (29): $[0\ -\ -]$; $[+\ +\ 0]$;
  $[-\ -\ -]$; $[+\ +\ 0]$;
  $[0\ -\ -]$; $[+\ +\ +]$;
- in the case (30): $[-\ -\ -]$; $[0\ +\ +]$;
  $[-\ -\ 0]$; $[0\ +\ +]$;
  $[-\ -\ 0]$; $[+\ +\ +]$;
- in the case (31): $[-\ -\ 0]$; $[+\ +\ 0]$;
  $[0\ -\ -]$; $[0\ +\ +]$.

Hence, the implication

$$\{m_i = 0 \;\Leftrightarrow\; a_{ii} = 0\} \tag{32}$$

holds in the cases (28) and (31) only. In the case (28) the derivative of g with respect to d_1 has the same sign of

$$d_1 - \delta_1 = d_1 - \{m_1 a_{33}/(m_3 a_{11})\}^{1/2},$$

hence (26) is equivalent to the positiveness of $g(\delta_1)$, the global minimum of $g(d_1)$ over $d_1 > 0$:

$$g(\delta_1) > 0.$$

It is easy to check that this property just amounts to (16).

In the case (29) (or (30)), g is strictly decreasing (increasing) with $d_1 > 0$, therefore (26) is equivalent to $g(+\infty) \geqq 0 \quad (\text{or } g(0) > 0)$, i.e. to

$$\tau \geqq 0,$$

being $\tau = \Delta + 2\{m_2 a_{22}(m_1 a_{11} + m_3 a_{33})\}^{1/2}$.

In the case (31) g is constant on the value τ, therefore (26) is equivalent to

$$\tau > 0.$$

Note that $m_{11} a_{11}$ and/or $m_3 a_{33}$ vanish in the cases (29) to (31) (see the sign patterns above); hence, according to (15) and (19), $\tau = t$ follows. Therefore the theorem is proved for the case (25).

Now, let (25) be false. It is useful to separate the following two cases:

$$a_{22}(m_1 + m_3) = m_2(a_{11} + a_{33}) = 0 \tag{33}$$

$$m_2 a_{22}(m_1 + m_3)(a_{11} + a_{33}) = 0 > a_{22}(m_1 + m_3) + m_2(a_{11} + a_{33}) \tag{34}$$

and to define

$$\theta = \Delta + 2\{m_1 m_3 a_{11} a_{33}\}^{1/2}.$$

In the case (33) the list of the sign patterns of the vectors a and m can be obtained from the one yet shown for the case (25) above, simply by putting 0 for the signs of both a_{22} and m_2 and adding the extra case

$$a = [\,0 \;\; - \;\; 0\,], \quad m = [\,0 \;\; + \;\; 0\,],$$

for the case (31). Again, the implication (32) holds in the cases (28) and (31) only. Moreover, it is easy the check that (24) is equivalent to

$$\beta \quad \begin{cases} > 0, \ \forall d_1 > 0, & \text{in the case (33);} \\ \geqq 0, \ \forall d_1 > 0, & \text{in the case (33),} \end{cases}$$

and that this simply means

$$\theta \quad \begin{cases} > 0, & \text{if (32) holds and (33) occurs;} \\ \geqq 0, & \text{otherwise.} \end{cases}$$

We may also note that from (33) or (34) it follows $\mathrm{m}_2 a_{22} = 0$ and/or $m_1 a_{11} = m_3 a_{33} = 0$, hence $\theta = t$ and therefore the theorem is proved also for the case where (25) does not hold. This completes the proof. □

Remark 1

From the basic assumption $A \in (NP)_0^+$, the non-negativeness of both α and γ follows. Moreover (21) and (22) show that the additional assumption

$$\Delta > 0,$$

i.e.

$$-2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{13}a_{32} > 0, \tag{35}$$

is sufficient to ensure that β is positive too, therefore that (24) holds, hence that A is D-stable. Condition (35) is just the sufficient condition for the D-stability of a matrix of third order, proposed by Johnson (1974), taking into account that this author considers positive stability. Obviously, there are matrices which do not verify (35) and yet are D-stable. Johnson (1974) gives an example; another example is

$$A = \begin{bmatrix} -1 & +2 & -0,5 \\ +0,4 & -1 & -1 \\ -0,2 & +13 & -1 \end{bmatrix}.$$

The matrix A is D-stable ($t \approx 11,09$), but does not verify (35), as in this case $\Delta = -0,2 < 0$.

Remark 2

As D-stability implies the usual stability of a square matrix, it is obvious that t cannot be greater than the value

$$T = (-a_{11} - a_{22} - a_{33})(m_1 + m_2 + m_3) + \det A \tag{36}$$

that $f(d)$ assumes for $d = [1,\ 1]$, i.e. the usual value whose positiveness must be checked in order to be sure that an $(NP)_0^+$ matrix is stable. Note that from (15) and (36) the relation

$$\begin{aligned} T - t \;=\; & \left\{(-m_1 a_{22})^{1/2} - (-m_2 a_{11})^{1/2}\right\}^2 + \left\{(-m_1 a_{33})^{1/2} - (-m_3 a_{11})^{1/2}\right\}^2 + \\ & + \left\{(-m_1 a_{33})^{1/2} - (-m_3 a_{22})^{1/2}\right\}^2 \end{aligned}$$

follows. As $(T - t)$ vanishes if and only if the vectors a and m are linearly dependent, the usual Routh-Hurwitz test for the stability of a real $(NP)_0^+$ matrix of order 3 is indeed also a test for the D-stability of A if and only if $a = \eta m$ for some $\eta < 0$.

Always for $n = 3$, Cross (1978) has given a finite test for the additive stability (A, of order 3, is additively stable if and only if $A \in (NP)_0^+$ and A is stable) and for the Volterra-Lyapunov stability. The reader must pay attention, as in this paper of Cross results on negative stability and results on positive stability are continuously mixed.

The case $n = 4$ is indeed more difficult than the case $n = 3$. The necessary and sufficient conditions given by Johnson (1974) are not numerically checkable. There are at least (as for as we know) two attempts to construct a finite criterion for D-stability with $n = 4$, but, in our opinion, these criteria are not completely satisfactory.

Moreover, it seems that the problem of characterizing D-stability for general nxm matrices is NP-hard (see Chen, Fan and Yu (1995)). The first paper to treat the characterization of D-stability for $n = 4$ is due to Kanovei and Logofet (1998); the second paper is due Impram, Johnson and Pavani (2005) and it is based on some results of Johnson and Tesi (1999). Both papers start from the classical Routh-Hurwitz criterion, but Kanovei and Logofet transform the problem into one of polynomial programming (an idea hidden in the paper of Johnson (1975)), whereas Impram and others consider chains of Hankel determinants.

Necessary conditions and some sufficient conditions for $n = 5$ have been examined by Burlakova (2009).

6. Conclusion

In the last Section we have seen that the problem of characterizing D-stability for matrices of order $n > 4$ has not (since now) been solved and that the question, from a numerical point of view, grows “exponentially” in difficulty.

Perhaps it would be better to concentrate the efforts in order to find new classes of matrices (beyond Metzler and Morishima matrices) for which stability and D-stability coincide. In this respect a good starting point could be the paper of Hershkowitz and Keller (2003).

References

Abed, E. H. (1986), Strong D-stability, Systems and Control Letters, 7, 207–212.

Arrow, J. (1973), Stability independent of adjustment speed; in G. Horwich and P. A. Samuelson (Eds), Trade, Stability and Macroeconomics - Essays in Honor of Lloyd A. Metzler, Academic Press, New York, 181–202.

Arrow, K. J. and McManus, M. (1958), A note on dynamic stability, Econometrica, 26, 448-454.

Ballantine, C. S. (1970), Stabilization by a diagonal matrix, Proceedings of the Am. Math. Soc., 25, 728–734.

Beavis, B. and Dobbs, I. M. (1990), Optimization and Stability Theory for Economic Analysis, Cambridge Univ. Press, Cambridge.

Berman, A. and Hershkowitz, D. (1983), Matrix diagonal stability and its implications, SIAM J. Alg. Disc. Meth. 4, 377–382.

Burlakova, L. A. (2009), Conditions of D-stability of the fifth-order matrices; in V. P. Gerdt, E. W. Mayr, E. V. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing, Springer, Berlin, 54-65.

Cain, B. E. (1976), Real, 3x3, D-stable matrices, Journal of Research of the National Bureau of Standards, B. Mathematical Sciences, vol 80 B, 75-77.

Carlson, D. and Johnson, C. R. (1974), Research problem, Linear and Multilinear Algebra, 2, 185.

Chen, J., Fan, M. K. H. and Yu, C. (1985), On D-stability and structured singular values, Systems Control Lett., 24, 19-24.

Cross, G. W. (1978), Three types of matrix stability, Linear Algebra and Its Appl., 20, 253-263.

Datta, B. N. (1978), Stability and D-stablity, Linear Algebra and Its Appl. 21, 253–263.

Enthoven, A. C. and Arrow, K. J. (1956), A theorem on expectations and the stability of equilibrium, Econometrica, 24, 288–293.

Fiedler, M. and Pták, V. (1962), On matrices with non-positive off-diagonal elements and positive principal minors, Czechoslovak Mathematical Journal, 12, 382-400.

Fiedler, M. and Pták, V. (1966), Some generalizations of positive definiteness and monotonicity, Numerische Mathematik, 9, 163–172.

Fisher, M. E. and Fuller, A. T. (1958), On the stabilization of matrices and the convergence of linear iterative processes, Proc. Cambridge Philos. Soc., 54,. 417-425.

Fisher, F. M. (1972), A simple proof of the Fisher-Fuller theorem, Proc. Cambridge Philos. Soc., 71, 523-525.

Gantmacher, F. R. (1966), Theorie des matrices, Dunod, Paris.

Giorgi, G. (2003), Stable and related matrices in economic theory, Control and Cybernetics, 32, 397-410.

Greenberg, H. G. and Maybee, J. S. (1981), Computer- Assisted Analysis and Model Simplification, Academic Press, New York.

Hahn, F. (1982), Stability; in K. J. Arrow, M. D. Intriligator (Eds.), Handbook of Mathematical Economics, vol. II, North-Holland, Amsterdam, 745-793.

Hershkowitz, D. (1992), Recent directions in matrix stability, Linear Algebra and Its Applications, 171, 161-186.

Hershkowitz, D. and Keller, N. (2003), Positivity of principal minors, sign symmetry and stability, Linear Algebra and Its Applications, 364, 105-124.

Impram, S., Johnson, R. and Pavani, R. (2005), The D-stability problem for 4x4 real matrices, Archivum Mathematicum (Brno), 41, 439-450.

Jeffries, C. and Klee, V. and Van Den Driessche, P. (1977), When is a matrix sign stable? Canadian J. of Math., 29, 315-326.

Johnson, C. R. (1974), Second, third, and fourth order D-stability, Journal of Research of the National Bureau of Standards, 78 B, 11-13.

Johnson, C. R. (1974 a), D-stability and real and complex quadratic forms, Linear Algebra and Its Appl., 9, 89-94.

Johnson, C. R. (1974 b), Sufficient conditions for D-stability, Journal of Econ. Theory, 9, 53-62.

Johnson, C. R. (1974 c), Hadamard product of matrices, Linear and Multilinear Algebra. 1, 295-307.

Johnson, C. R. (1975), A characterization of the nonlinearity of D-stability, J. of Math. Economics, 2, 87-91.

Johnson, R. and Tesi, A. (1999), On the D-stability problem for real matrices, Boll. Unione Mat. Ital., Sez. B, 2, 299–314.

Kafri, W. S. (2002), Robust D-stabilty, Applied Mathematics Letters, 15, 7-10.

Kanovei, G. V. and Logofet, D. O. (1998), D-stability of 4-by-4 matrices, Computational Mathematics and Mathematical Physics, 38, 1369-1374.

Kaszkurewicz, E. and Bhaya, A. (2000), Matrix Diagonal Stability in Systems and Computation, Birkhäuser, Boston.

Kemp, M. C. and Kimura, Y. (1978), Introduction to Mathematical Economics. Springer Verlag, New York.

Khalil, H. K. (1980), A new test for D-stability, Journal of Econ. Theory 23, 120-122.

Kimura, Y. (1981), A note on sufficient conditions for D-stability, J. of Math. Economics, 8, 113-120.

Locatelli, A. and Schiavoni, N. (2012), A necessary and sufficient condition for the stabilisation of a matrix and its principal submatrices, Linear Algebra and Its Appl., 436, 2311-2314.

Magnani, U. (1972), Su una congettura di D-stabilità di Arrow e McManus, Fascicoli dell'Istituto di Matematica Generale e Finanziaria, N. 48, a.a. 1972/73, Università di Pavia.

Magnani, U. (1990), A test of D-stability for real matrices of order 3 (extended abstract), Atti XIV Convegno A.M.A.S.E.S., Pescara, 13-15 Settembre 1990, 267-268.

Magnani, U. and Meriggi, M. R. (1981), Characterizations of K-matrices; in G. Castellani, P. Mazzoleni (Eds), Mathematical Programming and Its Economic Applications, F. Angeli, Milano, 535-547.

Maybee, J. and Quirk, J. (1969), Qualitative problems in matrix theory, SIAM Review, 11, 30-51.

McKenzie, L. W. (1960), Matrices with dominant diagonals and economic theory; in K. J. Arrow, S. Karlin and P. Suppes (Eds.), Mathematical Methods in the Social Sciences, Stanford Univ. Press, Stanford, 1960, 47-62.

Metzler, L. (1945), Stability of multiple markets: the Hicks conditions, Econometrica, 13, 277-292.

Morishima, M. (1952), On the laws of change of the price-system in an economy which contains complementary commodities, Osaka Economic Papers, 1, 101-113.

Morishima, M. (1970), A generalization of the gross subtitute system, Rev. of Econ. Studies, 37, 177-186.

Newman, P. K. (1959), Some notes on stability conditions, Review of Economic Studies, 27, 1-9.

Plemmons, R. J. (1977), M-matrix characterizations, I-non-singular M-matrices, Linear Algebra and Its Appl., 18, 175-188.

Poole, G. and Boullion, T. (1974), A survey on M-matrices, SIAM Rev. 16, 419-427.

Quirk, J. P. (1981), Qualitative stability of matrices and economic theory: a survey article; in H. J. Greenberg and J. S. Maybee (Eds.), Computer Assisted Analysis and Model Simplification, Academic Press, New York, 113-164.

Quirk, J. P. and Ruppert, R. (1965), Qualitative economics and the stability of equilibrium, Review of Econ. Studies, 32,. 311-326.

Quirk, J. P. and Saposnik, R. (1968), Introduction to General Equilibrium Theory and Welfare Economics, McGraw-Hill, New York.

Samuelson, P. A. (1944), The relation between Hicksian stability and true dynamic stability, Econometrica, 12, 256-257.

Samuelson, P. A. (1947), Foundations of Economic Analysis, Harvard Univ. Press, Cambridge, Mass.

Woods, J. E. (1978), Mathematical Economics. Topics in Multi-Sectoral Economics, Longman Group Ltd, London.