Nonnegative square matrices: irreducibility, reducibility, primitivity and some economic applications

Giorgio Giorgi
(University of Pavia)

# 175 (10-19)

Via San Felice, 5
I-27100 Pavia

economiaweb.unipv.it
Nonnegative Square Matrices: Irreducibility, Reducibility, Primitivity and Some Economic Applications

Giorgio Giorgi (*)

Abstract  We give an overview of the various characterizations of irreducibility, reducibility and primitivity for nonnegative square matrices. Some comments are made for the so-called “Gantmacher normal form” and for a turnpike theorem, due to Achmanov (1984). In addition, we discuss a proposal of Manara (1968) for the reducibility of a Sraffa model with joint production.

Key words and phrases  Irreducible and reducible matrices, Gantmacher normal form, turnpike theorem, Sraffa joint production model.


1. Introduction

Nonnegative matrices and nonnegative square matrices play a central role in many branches of applied mathematics and of several other sciences: we may mention probability theory (Markov chains), population models, iterative methods in numerical analysis, epidemiology, stability analysis, linear economic models, etc. Among the last ones, particularly important are Leontief models (input-output models), Sraffa models, Von Neumann balanced growth models, Stolper-Samuelson models of international trade, linear programming models, games theory, etc. In dealing with square matrices and with nonnegative square matrices one inevitably encounters the concepts of irreducibility, reducibility, primitivity and imprimitivity.

The aim of the present paper is to collect the basic properties and characterizations of irreducible, reducible, primitive and imprimitive nonnegative square matrices and to give some economics-oriented applications of the said concepts, together with some new results and properties.

The paper is organized as follows. In Section 2 we recall the various characterizations and properties of irreducible, reducible, primitive and imprimitive nonnegative square matrices. In Section 3 we deal with an application of the so-called Gantmacher normal form of a square nonnegative matrix. In Section 4, following Achmanov (1984), we give an economic application of primitive matrices in describing a turnpike theorem for a dynamic Leontief production model.

(*) Department of Economics and Management, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: giorgio.giorgi@unipv.it
In the final Section 5 we discuss a proposal of reducibility, due to Manara (1968), for a Sraffa model with joint production, i.e. for a pair \((A, B)\), with \(A\) and \(B\) nonnegative square matrices of the same order.

In this paper \(A\) is a (real) matrix of \(m\) rows and \(n\) columns, with \(a_{ij}\) on its \(i\)-th row \(A_i\) and on its \(j\)-th column \(A_j\). If \(m = n\) we speak of square matrices of order \(n\). The symbol \([0]\) is used to denote a zero vector or a zero matrix, whereas the notations \(A > [0]\), \(A \geq [0]\), \(A \geq [0]\) are used, respectively, to define a positive matrix, i.e. \(a_{ij} > 0\), \(\forall i, j\); a nonnegative matrix, i.e. \(a_{ij} \geq 0\), \(\forall i, j\); a semipositive matrix, i.e. \(A \geq [0]\), but \(A \neq [0]\). \(A < [0]\), \(A \leq [0]\), \(A \leq [0]\) are defined in a similar way. The same convention is used to compare vectors of \(\mathbb{R}^n\) with the zero vector \([0] \in \mathbb{R}^n\).

2. Reducibility, Irreducibility and Primitivity of Nonnegative Square Matrices

Usually, given a (real) square matrix \(A\) of order \(n \geq 2\), not necessarily nonnegative, the same is said to be reducible or decomposable or also non-connected, when it is possible, by permutation of some rows and the corresponding columns, to obtain the block-partition

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ [0] & A_{22} \end{bmatrix}
\]  \hspace{1cm} (1)

or equivalently

\[
A = \begin{bmatrix} A_{11} & [0] \\ A_{21} & A_{22} \end{bmatrix},
\]  \hspace{1cm} (2)

with \(A_{11}\) (and therefore also \(A_{22}\)) square blocks. In other words, it must exist a permutation matrix \(P\) or equivalently \(Q\), such that \(PAP^\top\) is in the form (1) or equivalently, such that \(QAQ^\top\) is in the form (2).

Indeed, it is easy to verify that if, by means of the said permutations, we get form (1), we get also form (2) and vice-versa. If such a permutation matrix \(P\) (or \(Q\)) does not exist, then \(A\) is said irreducible or indecomposable or connected. If \(A\) is reducible (respectively: irreducible), also \(A^\top\) is reducible (respectively: irreducible). Some authors (e.g. Marcus and Minc (1974)) introduce also the notion of partly decomposable (or partly reducible) matrices: the square matrix \(A\) of order \(n\) is partly reducible if and only if there exist two permutation matrices \(P\) and \(Q\) (\(Q\) not necessarily coinciding with \(P^\top\)) such that \(PAQ\) is in one of the forms (1) or (2). Clearly, any reducible matrix is partly reducible, but the converse does not hold. For example

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

is partly reducible but not reducible. If \(A\) is not partly reducible, we say that \(A\) is fully irreducible or fully indecomposable. Every fully irreducible matrix is irreducible. We shall not treat the above definitions.
We have the following basic result (see, e. g., Kemp and Kimura (1978), Nikaido (1968, 1970), Takayama (1985)).

**Theorem 1.** The square matrix $A$ of order $n$ is reducible if and only if there exists a partition $\{N_1; N_2\}$, $N_1 \neq \emptyset$, $N_2 \neq \emptyset$, of the set $N = \{1, \ldots, n\}$ such that

$$\{i \in N_1, j \in N_2\} \implies a_{ij} = 0.$$ 

Note the real number 0 (considered as a matrix of order 1) is considered an irreducible matrix. However, we shall not take into consideration this case, except otherwise stated. Obviously, $A = [0]$, of order $n > 1$, is reducible. We remark that an irreducible matrix $A$ cannot have a zero row or a zero column; in case $A$ is nonnegative and irreducible, then each row of $A$ and each column of $A$ is a semipositive vector.

Note that if $A$ is reducible, then for any positive integer $p$, the power $(A)^p$ is reducible. This is no longer true for irreducible matrices. Take, for example, the irreducible matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

We have that for any power of $A$, $(A)^p$, with $p$ even integer, it holds

$$(A)^p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

(When $p$ is odd we have $(A)^p = A$).

On the other hand, if for example $(A)^p > [0]$ for some integer $p$, surely $A$ is irreducible, as if $A$ would be reducible, then $(A)^p$ would be reducible, in contrast with the assumption $(A)^p > [0]$.


**Theorem 2 (Perron-Frobenius theorem, “strong version”).** Let $A$ be a square nonnegative irreducible matrix of order $n \geq 2$. Then:

3
1. $A$ has a positive characteristic root $\lambda^*(A)$, called also Frobenius root or dominant root of $A$, with which a positive (right) column eigenvector $x^*$ can be associated, i.e. it holds

$$Ax^* = \lambda^*(A)x^*, \quad x^* > [0].$$

Moreover, all eigenvectors associated with other real characteristic roots of $A$ are not semipositive. The same holds for (left) row eigenvectors $p^*$. In other words $x^* > [0]$ ($p^* > [0]$) are unique up to multiplication by positive scalars.

2. $\lambda^*(A)$ increases when any element of $A$ increases.

3. $\lambda^*(A)$ is a simple root of the characteristic equation.

4. $\lambda^*(A) \leq |\lambda|$, $\forall \lambda \neq \lambda^*(A)$, being $\lambda$ any other eigenvalue of $A$ (if $\lambda^*(A) < |\lambda|$, $\forall \lambda \neq \lambda^*(A)$, then $A$ is called “primitive”).

5. It holds $(\rho I - A)^{-1} > [0]$ if and only if $\rho > \lambda^*(A)$.

**Theorem 3 (Perron-Frobenius theorem, “weak version”).** Let $A$ be a square non-negative matrix of order $n \geq 2$. Then:

1. $A$ has a nonnegative characteristic root $\lambda^*(A)$, with which a semipositive (right) eigenvector $x^*$ can be associated.

2. $\lambda^*(A)$ does not decrease when an element of $A$ increases.

3. It holds $\lambda^*(A) \leq |\lambda|$, $\forall \lambda \neq \lambda^*(A)$, being $\lambda$ any other eigenvalue of $A$.

4. It holds $(\rho I - A)^{-1} \geq [0]$ if and only if $\rho > \lambda^*(A)$.

5. It holds $\lambda^*(A) = 0$ if and only if $(A)^n = [0].$

For generalizations of the Perron-Frobenius theorem to a pair of nonnegative matrices $(A, B)$, see Bapat, Olesky and van Driessche (1995), Fujimoto (1977), Giorgi (2004), Mangasarian (1971).


**Definition 1.** A (real) square matrix $C$ of order $n$ is called a $Z$-matrix or matrix belonging to the $Z$-class if $c_{ij} \leq 0$, $\forall i \neq j$.

We note that in Economic Analysis, if $C$ is a $Z$-matrix, then $-C$ is called Metzlerian matrix (see, e.g., Kemp and Kimura (1978), Murata (1977), Woods (1978)). We note also that it is always possible to express a $Z$-matrix $C$ in the form

$$C = \lambda I - A,$$

where $\lambda \in \mathbb{R}$ and $A \geq [0]$. Indeed, it is sufficient to consider

$$\lambda \geq \max_i \{c_{ii}\}$$
and choose the matrix

\[ A = \lambda I - C \]

which is a nonnegative matrix.

An important subset of \( Z \)-matrices is given by \( M \)-matrices (or \( K \)-matrices). A square \( Z \)-matrix \( C \) is an \( M \)-matrix if \( C \) verifies any one of the equivalent properties contained in the following theorem (therefore if \( C \) verifies any one of the said properties, then \( C \) will verify all the other properties of the theorem).

**Theorem 4.** Let the square matrix \( C \) of order \( n \) be a \( Z \)-matrix. Then the following properties are equivalent.

1) There exists a vector \( x \in \mathbb{R}^n \) such that

\[
\begin{align*}
Cx &> [0] \\
x &\geq [0].
\end{align*}
\]

2) There exists a vector \( x \in \mathbb{R}^n \) such that

\[
\begin{align*}
Cx &> [0] \\
x &> [0].
\end{align*}
\]

3) There exists a vector \( y \in \mathbb{R}^n, y \geq [0] \), such that

\[
\begin{align*}
Cx &= y \\
x &> [0].
\end{align*}
\]

4) For any \( y \in \mathbb{R}^n, y \geq [0] \), the system

\[
\begin{align*}
Cx &= y \\
x &\geq [0]
\end{align*}
\]

has a solution.

5) The matrix \( C \) verifies the Hawkins-Simon conditions:

\[
\begin{align*}
c_{11} > 0, & \quad \left| \begin{array}{cc} c_{11} & c_{12} \\
c_{21} & c_{22} \end{array} \right| > 0, & \quad \left| \begin{array}{ccc} c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33} \end{array} \right| > 0, & \cdots, |C| > 0.
\end{align*}
\]

6) The inverse \( C^{-1} \) exists and it holds \( C^{-1} \geq [0] \).

7) If \( C \) is expressed in the form

\[ C = \lambda I - A, \text{ with } A \geq [0], \]  \hspace{1cm} (3)

it holds

\[ \lambda > \lambda^*(A). \]
8) When $C$ is expressed as in (3), $C^{-1}$ is given by the following series expansion ("series of C. Neumann"):

$$C^{-1} = (\lambda I - A)^{-1} = \frac{1}{\lambda} (I - \frac{1}{\lambda} A)^{-1} = \frac{1}{\lambda} \left\{ I + \frac{1}{\lambda} A + \frac{1}{\lambda^2} (A)^2 + \frac{1}{\lambda^3} (A)^3 + \ldots \right\} = \sum_{k=0}^{+\infty} \frac{1}{\lambda^{k+1}} (A)^k. $$

9) If $C = I - A$, $A \geq [0]$, it results

$$\lim_{k \to +\infty} (A)^k = [0].$$

**Remark 1.** If $C$ is also irreducible, some of the previous tests can be reformulated in a slightly different form, e. g.,

- In 1) and in 2), instead of $Cx > [0]$ it holds $Cx \geq [0]$.
- In 3), instead of $y > [0]$ it holds $y \geq [0]$.
- In 6), instead of $C^{-1} \geq [0]$, it holds $C^{-1} > [0]$.

It is also easy to prove that if $C$ is irreducible and $x \in \mathbb{R}^n$ solves the system

$$Cx \geq [0], \quad x \geq [0]$$

or the system

$$Cx = y, \quad y \geq [0], \quad x \geq [0],$$

then it will hold $x > [0]$.


**Theorem 5.** Let $A \geq [0]$ be of order $n \geq 2$ and let $\lambda^*(A)$ be its Frobenius root. Then the following statements are equivalent.

(a) $A$ is irreducible.
(b) For any pair $(i, j)$ of indices, $1 \leq i, j \leq n$, there is a positive integer $k(i, j), k \leq n$, such that $(a_{ij})^k > 0$.
(c) $(I + A)^{n-1} > [0]$.
(d) $(I + A + (A)^2 + \ldots + (A)^{n-1}) > [0]$.
(e) $A + (A)^2 + \ldots + (A)^n > [0]$. 

6
Every positive right eigenvector of $A$ is a positive scalar multiple of $x^*$, right eigenvector corresponding to $\lambda^*(A)$. In other words, $A$ has exactly one, up to scalar multiplication, positive eigenvector and this eigenvector is associated to $\lambda^*(A)$.

For some $\mu > \lambda^*(A)$, the matrix $(\mu I - A)$ is nonsingular and it holds $(\mu I - A)^{-1} > [0].$

For every $\mu > \lambda^*(A)$, the matrix $(\mu I - A)$ is nonsingular and it holds $(\mu I - A)^{-1} > [0].$

Remark 2. Note that in characterization (b) of the previous theorem the power $k$ depends on the pair $(i, j)$, therefore it may vary with $(i, j)$. Sometimes a wrong characterization of irreducibility is given, in the sense that it is asserted that $A \geq [0]$ is irreducible if and only if there exists $k \geq 1$ such that $(A)^k > [0]$. Recall the example of the irreducible matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The equivalence between (c) and (d) is evident. The following equivalences

$$(d) \iff (a) \iff (e)$$

can be proved simply in the following way. A result of Wielandt (1950) (“Wielandt lemma”) states that if $A \geq [0]$ of order $n$, is irreducible then it holds

$$(I + A)^{n-1} x > [0], \forall x \geq [0].$$

Taking $x = e^i$, $i = 1, ..., n$, where $e^i$ is the $i$-th elementary vector of $\mathbb{R}^n$, we have

$$(I + A)^{n-1} > [0],$$
i. e. proposition (c). From this proposition we get immediately proposition (d):

$$(I + A + (A)^2 + ... + (A)^{n-1}) > [0].$$

Multiplying the first side of this last relation by $A$ and recalling that $A$ cannot have any zero line (being irreducible) we get proposition (e). The proof is completed by noting that if $A$ is reducible, then every its power is reducible and therefore propositions (d) and (e) cannot hold.

The proofs of assertions (g) and (h), for $\mu = 1$, can be sketched as follows. If $\lambda^*(A) < 1$ and $A \geq [0]$ is irreducible, then a classical result of the Perron-Frobenius theorem gives $(I - A)^{-1} > [0].$ Let $\lambda^*(A) < 1$ and let $(I - A)^{-1} > [0]$; we prove that the matrix $A \geq [0]$ is irreducible. Suppose that $A$ is reducible; then it can be transformed into a matrix

$$P A P^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where $A_{11}$ is a square submatrix of order $k$ and $A_{22}$ is a square submatrix of order $(n - k)$. Then

$$I - A = \begin{pmatrix} I - A_{11} & -A_{12} \\ 0 & I - A_{22} \end{pmatrix}.$$
Let us denote the following matrices:

\[ K = I - A_{11}, \quad F = -A_{12}, \quad H = I - A_{22}. \]

Then we can see that

\[
(I - A)^{-1} = \begin{pmatrix}
K^{-1}(I + FH^{-1}) & -K^{-1}FH^{-1} \\
0 & H^{-1}
\end{pmatrix}.
\]

This is a contradiction to the assumption \((I - A)^{-1} > [0]\).

Another useful characterization of irreducibility of \(A \geq [0]\) of order \(n > 1\) is related to graph theory. See, e.g., Bondy and Murty (1976), Harary, Norman and Cartwright (1966), Lovasz (1979), Stefani and Camiz (1996), Szyld (1985).

We begin to state the following basic result (see, e.g., Abraham-Frois and Berrebi (1976), Basilevski (1983), Nikaido (1968)).

**Theorem 6.** Let \(A \geq [0]\) of order \(n > 1\); then \(A\) is irreducible if and only if for each pair of indices \((i, j)\) it is possible to build a “chain” of indices \(h_1, h_2, ..., h_r, h_s\), such that

\[ a_{ih_1}a_{h_1h_2}...a_{h_rh_s}a_{hsj} > 0. \]

We recall that a (directed) graph is a pair \(G = (V, E)\), where \(V = \{v_1, v_2, ..., v_n\}\) is a finite set of nodes (or vertices) of the graph and \(E = \{e_1, e_2, ..., e_m\} \subset V \times V\) is a subset of couples of nodes (the edges of the graph). If \(E\) is a subset of ordered couples of nodes, the graph \(G\) is an oriented graph or directed graph. If \(G = (V, E)\) is a directed graph, a directed path in \(G\) is a sequence of nodes \((v_0, v_1, ..., v_k)\) such that \((v_i, v_{i+1}) \in E\) for each \(i = 0, 1, ..., k - 1\). We say that \(k\) is the length of the path. A non-directed path in \(G\) is a sequence of nodes \((v_0, v_1, ..., v_k)\) such that either \((v_i, v_{i+1})\) or \((v_{i+1}, v_i)\) belongs to \(E\) for each \(i = 0, 1, ..., k - 1\).

A directed graph is connected if for every pair of nodes \((v_i, v_j)\) in \(V\) there exists a non-directed path that joins them. A directed graph is strongly connected if for every pair of nodes \((v_i, v_j)\) in \(V\) there exists a directed path that joins them. The square matrix \(A \geq [0]\) of order \(n\) can be associated to a directed graph \(G\), with \(n\) nodes, having an edge from node \(i\) to node \(j\) if and only if \(a_{ij} > 0\). This graph is usually called adjacency graph of the matrix. We have the following basic result.

**Theorem 7.** The matrix \(A \geq [0]\) of order \(n\) is irreducible if and only if its adjacency graph \(G\) is strongly connected.

The last two characterizations of irreducibility of a square nonnegative matrix \(A\) (Theorems 6 and 7) have an interesting economic interpretation, when \(A\) is an input-output matrix of a Leontief model or is a production matrix of a Sraffa model (without joint production). If \(A\) is irreducible, every sector of the economy has to deliver its goods directly or indirectly to each of the other sectors. In the terminology of Sraffa, each good is a basic good (or basic commodity). See Abraham-Frois and Berrebi (1976), Bidard (2004), Giorgi and Magnani (1978), Kurz and

Other useful characterizations of irreducible nonnegative square matrices are given in Berman and Plemmons (1979). See also Achmanov (1984) and Bidard (2004).

**Theorem 8.** Let $A \geq [0]$ be a square matrix of order $n > 1$. The following conditions are equivalent.

(a) The matrix $A$ is irreducible.

(b) No eigenvector of $A$ has a zero element.

(c) The proposition \{There exists $\lambda > 0$ such that $\lambda x \geq Ax$, with $x \geq [0]$\} implies $x > [0]$.

**Remark 3.** Condition (b) is another form to express condition (f) of Theorem 5: all eigenvectors of $A$ have elements of the same sign, if associated to $\lambda^*(A)$; otherwise, if real, they have elements with mixed signs; if complex, they have no zero elements. Condition (c) was given by Nikaido (1968, 1970) who, however, characterizes the reducibility of a square matrix $A \geq [0] : A \geq [0]$ is reducible if and only if there exist a nonnegative real number $\lambda$ and an appropriate vector $x \geq [0]$, but not positive, such that $\lambda x \geq Ax$.

Another characterization of irreducibility of nonnegative square matrices is given by Gantmacher (1959) and Cherubino (1956, 1957). This characterization results from the fact that $A$ is irreducible if and only if $A^\top$ is irreducible.

**Theorem 9.** Let $A \geq [0]$ square of order $n$. A necessary and sufficient condition for $A$ to be irreducible is that its Frobenius root $\lambda^*(A)$ is simple and that $A$ and $A^\top$ possess eigenvectors corresponding to $\lambda^*(A)$ which are positive (i.e., left (row) eigenvectors and right (column) eigenvectors associated to $\lambda^*(A)$ are both positive).

This last characterization is quite useful in studying the existence of both equilibrium prices and levels of production in linear economic models, such as, for example, Leontief models and Sraffa models. See, e.g., Blakley and Gossling (1967), Giorgi and Magnani (1978).

Gantmacher (1959) gives also the following characterization of reducible nonnegative square matrices. Let us consider the determinants, of order $(n-1)$,

$$B_{11}(\lambda^*), B_{22}(\lambda^*), ..., B_{nn}(\lambda^*)$$

where $\lambda^* = \lambda^*(A)$ and $B_{ii}(\lambda^*)$ is the cofactor of the element $\lambda^* - a_{ii}$ in the matrix $(\lambda^* I - A)$:

$$\begin{bmatrix}
\lambda^* - a_{11} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & \lambda^* - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \lambda^* - a_{nn}
\end{bmatrix}.$$

**Theorem 10.** The matrix $A \geq [0]$ of order $n$ is reducible if and only if one of the relations

$$B_{ii}(\lambda^*) \geq 0, \ i = 1, 2, ..., n,$$
degenerates to an equality.

Another characterization of irreducible nonnegative square matrices is given by Fujimoto (1977) who applies to the linear case the definition of irreducible vector-valued functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, previously given by Morishima (1964) and by Morishima and Fujimoto (1974).

Theorem 11. Let $A \geq [0]$ square of order $n$; then $A$ is irreducible if and only if for any non-empty proper subset of indices $R \subset N = \{1, 2, ..., n\}$ the relations $x_i^1 = x_i^2$ for $i \in R$ and $x_j^1 < x_j^2$ for $j \notin R$ imply that there exists at least one $i \in R$ such that $(Ax^1)_i < (Ax^2)_i$.

It can be proved (see Schwarz (1966a,b), Berman and Plemmons (1979)) that if $A \geq [0]$ is irreducible and $B$ is any nonnegative square matrix of the same order of $A$, then $A + B$ is irreducible. The product of two irreducible matrices can be reducible. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

We have already remarked that a $k$-th power of an irreducible nonnegative matrix can be reducible. In this case (see Gantmacher (1959), Theorem 9 of Chapter III) $(A)^k$ is completely reducible, i. e. there exists a permutation matrix $P$ such that

$$P(A)^k P^\top = \begin{bmatrix} A_{11} & [0] & \cdots & [0] \\ [0] & A_{22} & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & A_{dd} \end{bmatrix},$$

where $A_{ii}$, $i = 1, ..., d$, are irreducible matrices having the same Frobenius root.

The product of two reducible nonnegative matrices $A, B$ can be irreducible. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

For other characterizations of nonnegative irreducible matrices and other considerations on this class of matrices, the reader is referred to Chakravarti (1975), Elsner (1976), Minc (1974a, 1974b), Pulman (1975), Schwarz (1966a, 1966b).

We have remarked, in the “strong” form of the Perron-Frobenius theorem (Theorem 2), that the related Frobenius (dominant) root $\lambda^*(A)$ is superior or equal to the absolute value of any other eigenvalue (or characteristic value) of $A$. Therefore irreducible nonnegative square
matrices can be classified by the number of eigenvalues whose absolute value is maximum. We have the following basic definition.

**Definition 2.** Let \( A \geq [0] \) be an irreducible square matrix of order \( n \), with \( \lambda^*(A) \) its Frobenius eigenvalue. If \( \lambda^*(A) > |\lambda| \), being \( \lambda \neq \lambda^*(A) \) any other eigenvalue of \( A \), then \( A \) is called a **primitive matrix** (or also an **acyclic matrix**). If \( A \) has \( h > 1 \) eigenvalues

\[
\lambda_1 = \lambda^*(A) = |\lambda_2| = \ldots = |\lambda_h|,
\]

then \( A \) is called an **imprimitive matrix** (or also a **cyclic matrix**). The value \( h \) is called the **index of imprimitivity** of \( A \). Therefore \( A \) is primitive if \( h = 1 \), imprimitive if \( h > 1 \).

The following two results are due to Wielandt (1950).

**Theorem 12.** Let \( A \) be an irreducible nonnegative matrix of order \( n \) with \( \lambda^*(A) = \lambda_1 \) as its Frobenius root and with \( h \) as its index of imprimitivity. Let \( \lambda_2, \ldots, \lambda_h \) be the distinct eigenvalues of \( A \), with \( |\lambda_i| = \lambda^*(A) \), \( i = 2, \ldots, h \). Then \( \lambda_1, \lambda_2, \ldots, \lambda_h \) are precisely the solutions of the equation \( \lambda^h - (\lambda^*(A))^h = 0 \).

**Theorem 13.** The spectrum of an irreducible matrix \( A \geq [0] \) of index \( h \) is invariant under a rotation of \( 2\pi/h \), but not through a positive angle smaller than \( 2\pi/h \).

We have seen that a nonnegative irreducible matrix can have a power not irreducible and that \( A \geq [0] \) of order \( n \) is irreducible if and only if \( (I + A)^{n-1} > [0] \). We now see that primitive matrices can be characterized by their powers. The following result is due to Frobenius (1912); see also Wielandt (1950) and Herstein (1954).

**Theorem 14.** A necessary and sufficient condition for an irreducible square matrix \( A \geq [0] \) to be primitive is that \( (A)^m \) is positive for some positive integer \( m \).

It turns out that a **positive square matrix** \( A \) is primitive. It must be stressed that the result of Theorem 14 is not obvious, as we have already remarked that the power of an irreducible matrix can be reducible. Wielandt (1950) has established, without proof, that, given \( A \geq [0] \) primitive of order \( n \), if we define \( \gamma(A) \) as the smallest positive integer \( m \) such that \( (A)^m > [0] \), then we have

\[
\gamma(A) \leq n^2 - 2n + 2 = (n - 1)^2 + 1.
\]

This "minimal value" \( \gamma(A) \) is also called **index of primitivity** of the primitive matrix \( A \).

**Theorem 15.** Let \( A \geq [0] \) be irreducible of order \( n \). Then the following statements are equivalent.

(a) \( A \) is primitive.

(b) \( (A)^m > [0] \) for some positive integer \( m \).

(c) \( (A)^m > [0] \) for all \( m \geq n^2 - 2n + 2 \).

(d) \( (A)^{n^2-2n+2} > [0] \).

(e) \( \lambda^*(A) = \lim_{k \to \infty} \left[ (a_{ij})^k \right]^{1/k} \), where here \( (a_{ij})^k \) denotes entries in \( (A)^k \).
For a proof of (d) in Theorem 15 see, e. g., Holladay and Varga (1958). For (e) see Meyer (2000), Marcus and Minc (1974).

It must be noted that if $A \geq [0]$ is primitive and $B \geq [0]$ is of the same order of $A$, then $A + B$ is primitive. Schwarz (1966b) gives an interesting (and new) characterization of irreducibility for nonnegative matrices in terms of primitivity.

**Theorem 16.** A nonnegative square matrix $A$ of order $n$ is irreducible if and only if $A + (A)^2$ is primitive. Equivalently: $A \geq [0]$ is irreducible if and only if $I + A$ is primitive.

We remark that the product of two primitive matrices need not be irreducible (and hence need not be primitive). Take for example the primitive matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$  

We have

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Schwarz (1966b) shows that this case cannot occur if the matrices $A, B$ commute.

The following results, due to Frobenius (1912) and better specified subsequently (see, e. g., Fiedler (1986)) show a connection between the spectral properties of an irreducible nonnegative matrix and the imprimitivity of the same.

**Theorem 17.** Let $A \geq [0]$ be an irreducible imprimitive matrix of order $n$ with $h > 1$ as its index of imprimitivity. Then the following properties of $A$ are equivalent.

(i) There exists a permutation matrix $P$ such that $PAP^\top$ has the form

$$PAP^\top = \begin{bmatrix}
0 & A_{12} & [0] & \cdots & [0] & [0] \\
0 & [0] & A_{23} & \cdots & [0] & [0] \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
[0] & [0] & [0] & \cdots & [0] & A_{h-1,h} \\
A_{h1} & [0] & [0] & \cdots & [0] & [0]
\end{bmatrix} \quad (4)$$

where the diagonal blocks are zero square matrices, and there is no permutation matrix which puts $A$ into an analogous form having more than $h$ block rows.

(ii) If

$$(-1)^n\lambda^n + k_{n_1}\lambda^{n_1} + k_{n_2}\lambda^{n_2} + \ldots + k_{n_s}\lambda^{n_s}$$

is the characteristic polynomial of the matrix $A$, and

$$k_{n_1} \neq 0, \ k_{n_2} \neq 0, \ldots, \ k_{n_s} \neq 0, \ n > n_1 > \ldots > n_s \geq 0,$$

12
then the greatest common divisor of the numbers
\[ n - n_1, \; n_1 - n_2, \ldots, n_{s-1} - n_s \]
is \( h \).

A first consequence of the previous result is the following theorem.

**Theorem 18.** If an irreducible matrix \( A \succeq [0] \) has a positive trace, i.e., has at least one element \( a_{ii} > 0 \), then \( A \) is primitive.

This result was also discovered by Solow (1952). The same result implies that if the irreducible matrix \( A \succeq [0] \) is imprimitive, then \( \text{trace}(A) = 0 \), in other words the sum of all its eigenvalues is zero.

**Remark 4.** Some authors call (4) the Frobenius form of \( A \). Theorem 18 confirms once more that \( A \succ [0] \) (square) is primitive. Moreover, it is evident that if \( A \succeq [0] \) is imprimitive, then \( \text{trace}(A) = 0 \); in other words the sum of all its eigenvalues is zero.

\( A \text{ is primitive} \) if and only if there exists a partition of \( N = \{1, 2, \ldots, n\} \) into nonempty subsets \( \Gamma_1, \Gamma_2, \ldots, \Gamma_h, \Gamma_{\ell} \cap \Gamma_k = \emptyset \), for \( \ell \neq k \), such that if \( a_{ij} > 0 \), \( i \in \Gamma_{k-1} \), then \( j \in \Gamma_k \), where \( \Gamma_0 \) is regarded as \( \Gamma_h \).

Nikaido (1968) gives an economic interpretation of the above characterization. See also Solow (1952) and Takayama (1985).

A result of Herstein (1954) says that if \( A \succeq [0] \) is primitive, then \( (A)^k \) is primitive (and hence irreducible) for every positive integer \( k \). Moreover, if \( A \succeq [0] \), of order \( n \), is primitive with \( a_{ii} > 0 \), \( \forall i = 1, \ldots, n \), then \( (A)^{n-1} > [0] \). Holladay and Varga (1958) have proved, as a special case of a more general theorem (see what previously asserted on the index of primitivity), that if at least one main diagonal element of the irreducible matrix \( A \succeq [0] \) is positive (and therefore \( A \) is primitive), then \( (A)^{2n-2} > [0] \).

Cherubino (1956, 1957) has proved that if \( A \succeq [0] \), square of order \( n \), has its powers
\[ A, \; (A)^2, \; (A)^3, \ldots, (A)^n \]
all irreducible, then $A$ is primitive. Therefore we have the following nice characterization of primitive matrices, subsequently rediscovered by Fujimoto (2004) and by Fujimoto and Ekuni (2004).

**Theorem 19.** Let $A \geq [0]$ be irreducible of order $n$. Then $A$ is primitive if and only if its powers $(A)^2, (A)^3, \ldots, (A)^n$, are irreducible matrices.

Other properties of powers of nonnegative primitive matrices, in particular the so-called ergodic theorem, will be recalled in Section 3. Here we anticipate that if $A$ is a nonnegative primitive matrix, then the limit

$$ L = \lim_{n \to \infty} \left( \frac{1}{\lambda^*(A)} A \right)^n $$

exists and is a positive matrix. Furthermore, if $x$ and $y$ are, respectively, left and right positive eigenvectors of $A$ corresponding to the Frobenius root $\lambda^*(A)$ and scaled so that $yx = 1$, then $L = xy$.

Finally, we point out a result of Debreu and Herstein (1953) which is a corollary of a more general result on the convergence of the powers of a nonnegative square matrix (see also Bapat and Raghavan (1977)):

- Let $A \geq [0]$ be an irreducible square matrix of order $n$ and let $\lambda^*(A) = 1$. Then the sequence $(A)^k$ converges to some matrix $B$ if and only if $A$ is primitive.

3. An Application of the Gantmacher Normal Form to the Perron-Frobenius Theorem

Let us suppose that a square matrix $A$ is decomposable; if in (1) or in (2) the block $A_{11}$ or the block $A_{22}$ (or both) is itself decomposable, it is possible, by means of a suitable permutation matrix $P$, to arrive to a form of the following type:

$$ PAP^\top = \begin{bmatrix} A_{11} & [0] & \cdots & [0] \\ A_{21} & A_{22} & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}, \quad (5) $$

with $A_{jj}$ square, $j = 1, \ldots, k$. This form is also called quasi-triangular form. If in (5) we have $A_{ij} = [0]$ for every $i, j$ with $i > j$, we have a quasi-diagonal form:

$$ PAP^\top = \begin{bmatrix} A_{11} & [0] & \cdots & [0] \\ [0] & A_{22} & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & A_{kk} \end{bmatrix}. $$

In this case we say that $A$ is completely reducible or completely decomposable.
On the other hand, a nonnegative square matrix $A$ can have a positive eigenvalue even if $A$ is reducible. For example, take

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. $$

We have $\lambda^*(A) = 1$ and

$$x^* = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \quad \alpha > 0$$

is the associated (column) Frobenius eigenvector.

If we take

$$A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we have $\lambda^*(A) = 1$ and

$$x^* = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta > 0. $$

In this case we have two linearly independent positive eigenvectors.

If we consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

we have $\lambda^*(A) = 1$, however $A$ has not column positive eigenvectors associated to $\lambda^*(A)$, being

$$x^* = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad \alpha > 0.$$

We can have the same situation also with $\lambda^*(A) = 0$; for example with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we have $\lambda^*(A) = 0$ (algebraic multiplicity =2) and

$$x^* = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad \alpha > 0.$$

Note that, in this case, $(A)^2 = [0] = (A)^n$ for every $n$. Obviously, with $A = [0]$ we have $\lambda^*(A) = 0$ and $x^* > [0]$.

In order to deepen these questions it is useful to introduce the so-called Gantmacher normal form (see Gantmacher (1959)).
Theorem 20. Let \( A \geq [0] \) be of order \( n \geq 2 \). Then, there exists a permutation matrix \( P \) such that

\[
PAP^\top = \begin{bmatrix}
  A_{11} & [0] & \cdots & [0] & [0] & \cdots & [0] \\
  [0] & A_{22} & \cdots & [0] & [0] & \cdots & [0] \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  [0] & [0] & \cdots & A_{gg} & [0] & \cdots & [0] \\
  A_{h1} & A_{h2} & \cdots & A_{hg} & A_{hh} & \cdots & [0] \\
  A_{i1} & A_{i2} & \cdots & A_{ig} & A_{ii} & \cdots & [0] \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  A_{s1} & A_{s2} & \cdots & A_{sg} & A_{sh} & A_{si} & \cdots & A_{ss}
\end{bmatrix}
\]

where:

\( a) \) Each “principal block” \( A_{11}, A_{22}, \ldots, A_{ss} \) is a square irreducible matrix of order \( k_j \geq 2 \), \( j = 1, \ldots, s \), or a scalar, possibly zero.

\( b) \) Each “stripe” staying on the left of the principal blocks \( A_{hh}, A_{ii}, \ldots, A_{ss} \) is nonzero, i.e.

\[
\begin{bmatrix}
  A_{h1}, & A_{h2}, & \ldots, & A_{hg} \\
  A_{i1}, & A_{i2}, & \ldots, & A_{ih} \\
  \vdots, & \vdots, & \ddots, & \vdots \\
  A_{s1}, & A_{s2}, & \ldots, & A_{ss-1}
\end{bmatrix} \geq [0].
\]

It is worth to remark that:

1) If in the above form we have \( g = s \), then

\[
PAP^\top = \begin{bmatrix}
  A_{11} & [0] & \cdots & [0] \\
  [0] & A_{22} & \cdots & [0] \\
  \vdots & \vdots & \ddots & \vdots \\
  [0] & [0] & \cdots & A_{gg}
\end{bmatrix},
\]

i.e. \( A \) is completely reducible; if \( g < s \), then \( A \) is in the quasi-triangular form. The blocks \( A_{11}, \ldots, A_{gg} \) are called “isolated blocks”.

2) If \( s = g = 1 \), then

\[
PAP^\top = A_{11},
\]

i.e. \( A \) is irreducible.

3) The Gantmacher normal form is unique, up to the possibility of:

\( a) \) permuting the isolated blocks \( A_{11}, \ldots, A_{gg} \);

\( b) \) permuting the subsequent principal blocks \( A_{hh}, A_{ii}, \ldots, A_{ss} \);

\( c) \) permuting, within the \( s \) principal blocks, the rows and corresponding columns of the blocks.

The operation sub \( b) \) is not always possible.
It is more convenient, above all for certain economic applications, to introduce a Gantmacher normal form which is, so to speak, a transpose of the form of Theorem 20, i.e., with $Q$ a suitable permutation matrix, the form

$$QAQ^\top = \begin{bmatrix}
A_{11} & [0] & \cdots & [0] & A_{1h} & \cdots & A_{1s} \\
[0] & A_{22} & \cdots & [0] & A_{2h} & \cdots & A_{2s} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
[0] & [0] & \cdots & A_{gg} & A_{gh} & \cdots & A_{gs} \\
[0] & [0] & \cdots & [0] & A_{hh} & \cdots & A_{hr} & A_{hs} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
[0] & [0] & \cdots & [0] & [0] & \cdots & [0] & \cdots & A_{rr} & A_{rs} \\
[0] & [0] & \cdots & [0] & [0] & \cdots & [0] & \cdots & [0] & A_{ss}
\end{bmatrix},$$

where properties similar to the ones previously introduced for the form of Theorem 20 hold.

Every square matrix $A \geq [0]$ admits both forms, but in them not always the pairs $(g, s)$ are the same and not always the principal blocks coincide. The Gantmacher normal form of nonnegative square matrices, has been considered in various economic applications, with reference to the study of the existence of equilibrium solutions, referred to activity levels or referred to prices, in linear models. See, e.g., Dantzig (1955), Giorgi and Magnani (1978), Krause (1981), Lippi (1979), Szyld (1985), Varri (1979), Zaghini (1967).

We first remark that $\lambda \in \mathbb{C}$ is an eigenvalue of $A \geq [0]$ if and only if $\lambda$ is an eigenvalue of at least one of the $s$ principal blocks of its Gantmacher normal form. In other words, the spectrum of $A$ is given by the union of the spectra of the principal blocks $A_{11}, \ldots, A_{ss}$. With reference to form (6), Gantmacher (1959) proves the following basic result.

**Theorem 21.** Let $A \geq [0]$ of order $n \geq 2$ be transformed into form (6). There is a positive (column) eigenvector $x^* > [0]$ corresponding to the Frobenius eigenvalue $\lambda^*(A)$ if and only if:

(i) Each of the isolated blocks $A_{11}, A_{22}, \ldots, A_{gg}$ in the Gantmacher normal form has $\lambda^*(A)$ as its eigenvalue.

(ii) When $s > g$ none of the matrices $A_{hh}, A_{ii}, \ldots, A_{ss}$ possesses this property, i.e.

$$\lambda^*(A_{hh}) < \lambda^*(A), \quad \lambda^*(A_{ii}) < \lambda^*(A), \ldots, \lambda^*(A_{ss}) < \lambda^*(A).$$

This theorem is important, e.g., to establish the existence of solutions for the so-called “standard system” of a Sraffa simple production model (i.e., with no joint production) where $A$ is reducible: in other words, there are also “non-basic commodities”, in the terminology of Sraffa (1960).

**Proof of Theorem 21.** We now prove in a simple way the above theorem, making use of the basic facts of the Perron-Frobenius theorem (strong version) and of the basic theory of $K$-matrices (or $M$-matrices). In order to avoid non interesting economic situations, we make the assumption that every row of $A$ is semipositive:

$$A_i \geq [0], \quad i = 1, \ldots, n,$$
also to exclude that some of the isolated blocks \( A_{11}, \ldots, A_{gg} \) is a zero scalar. Therefore, the isolated blocks \( A_{11}, \ldots, A_{gg} \) are semipositive irreducible matrices. We take into consideration the system
\[
\begin{aligned}
\begin{cases}
Ax = \lambda x \\
\lambda > 0, \ x > [0]
\end{cases}
\end{aligned}
\]  
with \( A \) reduced to its Gantmacher normal form \( (6) \). We rewrite the equations of \( (7) \) putting into evidence the blocks of the related Gantmacher normal form:
\[
\begin{bmatrix}
A_{11} & [0] & \cdots & [0] & [0] & \cdots & [0] \\
[0] & A_{22} & \cdots & [0] & [0] & \cdots & [0] \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
[0] & [0] & \cdots & A_{gg} & [0] & \cdots & [0] \\
A_{h1} & A_{h2} & \cdots & A_{hg} & A_{hh} & \cdots & [0] \\
A_{i1} & A_{i2} & \cdots & A_{ig} & A_{ih} & \cdots & [0] \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{s1} & A_{s2} & \cdots & A_{sg} & A_{sh} & A_{si} & \cdots & A_{ss}
\end{bmatrix}
\begin{bmatrix}
x^1 \\
x^2 \\
\vdots \\
x^g \\
x^h \\
x^i \\
\vdots \\
x^s
\end{bmatrix}
= \begin{bmatrix}
\lambda x^1 \\
\lambda x^2 \\
\vdots \\
\lambda x^g \\
\lambda x^h \\
\lambda x^i \\
\vdots \\
\lambda x^s
\end{bmatrix}.
\]

We recall that each “stripe” of blocks on the left of the principal blocks \( A_{hh}, \ldots, A_{ss} \) is not zero:
\[
A_{[t]} = [A_{t1}, A_{t2}, \ldots, A_{t,t-1}] \neq [0], \quad \forall t \in \{h, i, \ldots, s\}.
\]

We decompose this system into \( s \) subsystems. The first \( g \) subsystems are of the type
\[
\begin{aligned}
\begin{cases}
A_t x^t = \lambda x^t \\
\lambda > 0, \ x^t > [0]
\end{cases}
\end{aligned},
\]

\( t \in \{1, 2, \ldots, g\} \).

We recall that each block \( A_t, t \in \{1, 2, \ldots, g\} \) is (semipositive) irreducible and therefore we can apply to the said blocks the Perron-Frobenius theorem, in its “strong” version: \( \lambda \) must coincide with the Frobenius root of \( A_t, t \in \{1, 2, \ldots, g\} \), and therefore it must hold
\[
\lambda = \lambda^*(A_{11}) = \lambda^*(A_{22}) = \ldots = \lambda^*(A_{gg}).
\]

If \( g < s \), the other \( (s - g) \) subsystems are of the type
\[
\begin{aligned}
\begin{cases}
A_{[t]} \begin{bmatrix}
x^1 \\
\vdots \\
x^g \\
\vdots \\
x^{t-1}
\end{bmatrix} + A_t x^t = \lambda x^t \\
\lambda > 0, \ x^t > [0], \ t \in \{h, \ldots, s\}
\end{cases}
\end{aligned},
\]
i. e.

\[
\begin{cases}
(\lambda I - A_t)x^t = A_{[t]}x^{[t-1]} \\
\lambda > 0, \ x^t > [0], \ t \in \{h, ..., s\},
\end{cases}
\]

where

\[
x^{[t-1]} = \begin{bmatrix}
x^1 \\
\vdots \\
x^g \\
x^h \\
\vdots \\
x^{t-1}
\end{bmatrix}, \ t \in \{h, ..., s\}.
\]

Now, \((\lambda I - A_t)\) is, for every \(t \in \{h, ..., s\}\), a Z-matrix, with every block \(A_t\) irreducible. If we want that there exists a vector \(x^t > [0]\) such that for this Z-matrix it holds

\[(\lambda I - A_t)x^t = A_{[t]}x^{[t-1]} \geq [0],\]

the matrix \((\lambda I - A_t)\) must be a K-matrix (called also M-matrix). But then it must hold

\[\lambda > \lambda^*(A_t), \ \forall t \in \{h, ..., s\}.\]

Then \(x^t\) is found by means of the inverse of \((\lambda I - A_t)\), inverse which exists and is positive (every block \(A_t\) is irreducible):

\[x^t = (\lambda I - A_t)^{-1}A_{[t]}x^{[t-1]} > [0].\]

\[\square\]

4. An Economic Application of Primitive Matrices: a Turnpike Theorem


An interesting result in economic theory is the “turnpike theorem” of Morishima (Morishima (1961, 1964)) for a Leontief-von Neumann model. In this result the assumption of primitivity of the input-output matrix is made, in order to get the desired theorem. In the present section we follow the simpler and more mathematically compact approach of Achmanov (1984) in order to obtain a similar result for a dynamic Leontief model. See also the interesting papers of Berezneva (1977), and of Dana, Florenzano, Le Van and Levy (1989 a,b).
Let us consider a planning interval of $T$ periods of time. At the initial time the existing stock of goods $x^0 = (x_0^0, x_0^1, \ldots, x_0^n) > [0]$ is used in order to produce the vector of goods $x^1 = (x_1^0, x_1^1, \ldots, x_1^n)$. This last vector is then used as input in order to produce the vector of goods $x^2$, and so on. The choice of the vector $x^{t+1}$ produced from $x^t$ is not unique, in the sense that the sequence $\{x^1, x^2, \ldots, x^T\}$ is only one of the possible trajectories that describe the evolution of the dynamic Leontief system. The choice of a trajectory indeed may be subject to several different requirements. In order to precise our treatment, we make the assumption that the economic function to be maximized has the form

$$c_1x_1^T + c_2x_2^T + \ldots + c_nx_n^T,$$

where the vector $c = [c_1, c_2, \ldots, c_n] \geq [0]$ describes, for example, the prices of the produced goods at the final period $T$ of the production program. We therefore consider the following linear programming problem

$$\max \left\{ \begin{array}{ll}
  cx^T \\
  Ax^1 \leq x^0 \\
  Ax^{t+1} \leq x^t, \ t = 1, 2, \ldots, T - 1 \\
  x^t \geq [0], \ t = 1, 2, \ldots, T,
\end{array} \right.$$

where $A \geq [0]$ is the input-output matrix, of order $n$.

In the present section, in order to avoid confusion with the final period of time $T$, the notations $x'$ and $A'$ will be used for the transpose of, respectively, $x$ and $A$. Moreover, let

- $x_A$ denotes the right Frobenius eigenvector of $A$;
- $p_A$ denotes the left Frobenius eigenvector of $A$;
- $\lambda_A^*$ denotes the Frobenius root of $A$ and $A'$.

Then we have

$$Ax_A = \lambda_A^* x_A; \quad p_A A = \lambda_A^* p_A.$$

Following Achmanov (1984), we recall an important result concerning primitive (nonnegative irreducible) matrices: the so-called “ergodic theorem”.

**Theorem 22. (Ergodic theorem).** Let $A \geq [0]$ of order $n$ be a primitive matrix and let $x \geq [0]$ be a vector of $\mathbb{R}_n^+$. It holds

$$\lim_{t \to +\infty} \left( \frac{1}{\lambda_A^*} A \right)^t = \mu x_A,$$

where

$$\mu = \frac{p_A x}{p_A x_A}.$$

We have already considered in Section 1 this type of result. The name “ergodic theorem” comes from Probability Theory: the theorem says, in a synthetic way, that the sequence

$$\left( \frac{1}{\lambda_A^*} A \right)^t x, \ t = 1, 2, \ldots$$
is convergent when \( x \in \mathbb{R}^n_+ \).

**Definition 3.** Let \( \bar{x} \in \mathbb{R}^n \) and \( \varepsilon > 0 \). The set

\[
C_\varepsilon(\bar{x}) = \{ x : x \in \mathbb{R}^n, \exists \lambda > 0 \text{ such that } \|\lambda x - \bar{x}\| < \varepsilon \}
\]

is called \( \varepsilon \)-conic neighborhood of the vector \( \bar{x} \).

It is then possible to state the ergodic theorem in terms of the \( \varepsilon \)-conic neighborhood.

**Theorem 23.** Let \( A \geq [0] \) be a primitive matrix of order \( n \) and let \( x \geq [0] \) any nonnegative vector of \( \mathbb{R}^n_+ \). Then there exists for any \( \varepsilon > 0 \) a natural number \( T(\varepsilon, x) \) such that the vector \( x^t = (A)^t x \) belongs, for \( t > T(\varepsilon, x) \) to the \( \varepsilon \)-conic neighborhood \( C_\varepsilon(x_A) \) of the Frobenius eigenvector \( x_A \).

The Achmanov turnpike theorem for the maximization problem (8) is the following one.

**Theorem 24 (Turnpike theorem).** We suppose that in the dynamic Leontief model described by (8) the input-output matrix \( A \geq [0] \), of order \( n \), is primitive, that the vector of initial stocks \( x^0 \) is positive, and that the vector of prices \( c \) is semipositive. Then, there exists, for any \( \varepsilon > 0 \), two numbers \( T_1(\varepsilon) \) and \( T_2(\varepsilon) \), such that if the planning period \( T \) is sufficiently large, i.e. \( T > T_1(\varepsilon) + T_2(\varepsilon) \), then it holds \( \bar{x}^t \in C_\varepsilon(x_A) \), for \( t \in [T_1(\varepsilon), T - T_2(\varepsilon)] \), for any optimal trajectory \( \{\bar{x}^t\} \), solution of problem (8).

**Remark 5.** The economic meaning of the above result is the following one: if the planning period is sufficiently large, then all optimal trajectories of problem (8) have the property that for those periods indexed by \( t \), with \( T_1(\varepsilon) \leq t \leq T - T_2(\varepsilon) \), the direction expressed by the activity vector \( \bar{x}^t \) is “close” to the direction expressed by the Frobenius eigenvector \( x_A \). It must be observed that the number of these “good” periods is proportional to the planning interval \( T \), as \( T_1(\varepsilon) \) and \( T_2(\varepsilon) \) are independent from \( T \). It follows also that if the optimal trajectory \( \{\bar{x}^t\} \) is not available, the growth trajectory is “close” to the optimal trajectory, if the vectors \( x^t \) are chosen in such a way that the quantity (Radner’s distance or angular distance; see Radner (1961))

\[
\left\| \frac{x^t}{\|x^t\|} - \frac{x_A}{\|x_A\|} \right\|
\]

is the least possible.

In order to prove Theorem 24 we need some lemmas. First, we state the dual problem of (8). The vector of variables in (8) is of order \( nT \), therefore we put

\[
x = (x^1, x^2, ..., x^T)
\]

where each \( x^t \) is an \( n \)-dimensional vector. The right-hand side vector of constraints is \( b = (x^0, [0], ..., [0]) \) and the objective coefficients vector is \( c^* = ([0], [0], ..., c) \), where \([0] \in \mathbb{R}^n \). We can rewrite problem (8) in the form

\[
\begin{align*}
\max & \quad c^* x \\
R x & \leq b, \\
x & \geq [0],
\end{align*}
\]
where

\[ R = \begin{bmatrix} A & [0] & [0] & \cdots & [0] & [0] \\ -I & A & [0] & \cdots & [0] & [0] \\ [0] & -I & A & \cdots & [0] & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [0] & [0] & [0] & \cdots & A & [0] \\ [0] & [0] & [0] & \cdots & -I & A \end{bmatrix}. \]

Let \( p = (p^1, p^2, \ldots, p^T) \) be the vector of dual variables (of order \( nT \)). The dual problem of (8) is then written in the form

\[
\begin{aligned}
\min & \quad b^T p \\
\text{s.t.} & \quad R^T p \geq c^*, \\
& \quad p \geq [0],
\end{aligned}
\]

or also

\[
\begin{aligned}
\min & \quad x_0^T p^1 \\
\text{s.t.} & \quad A^T p^t - p^{t+1} \geq [0], \ t = 1, \ldots, T - 1, \\
& \quad A^T p^T \geq c, \\
& \quad p^t \geq [0], \ t = 1, 2, \ldots, T. 
\end{aligned}
\]

(9)

**Lemma 1.** The \( \varepsilon \)-conic neighborhood of vector \( \bar{x} \) is a convex cone.

**Proof.** It is sufficient to verify that \( \lambda x^1 \in C_\varepsilon(\bar{x}) \), for any \( \lambda > 0 \) and that \( x^1 + x^2 \in C_\varepsilon(\bar{x}) \), for any \( x^1, x^2 \in C_\varepsilon(\bar{x}) \). The first assertion is evident. The second assertion says that there exist numbers \( \lambda > 0, \mu > 0 \) such that \( \|\lambda x^1 - \bar{x}\| < \varepsilon, \|\mu x^2 - \bar{x}\| < \varepsilon \). We prove that

\[
\|\gamma (x^1 + x^2) - \bar{x}\| < \varepsilon,
\]

where \( \gamma = \lambda \mu/(\lambda + \mu) \). Indeed,

\[
\|\gamma (x^1 + x^2) - \bar{x}\| = \left\| \gamma (x^1 + x^2) - \frac{\mu}{\lambda + \mu} \bar{x} - \frac{\lambda}{\lambda + \mu} \bar{x} \right\| = \left\| \frac{\mu}{\lambda + \mu} (\lambda x^1 - \bar{x}) + \frac{\lambda}{\lambda + \mu} (\mu x^2 - \bar{x}) \right\| \leq \frac{\mu}{\lambda + \mu} \|\lambda x^1 - \bar{x}\| + \frac{\lambda}{\lambda + \mu} \|\mu x^2 - \bar{x}\| < \varepsilon.
\]

The following lemma is a refinement of the thesis of Theorem 23.

**Lemma 2.** Let \( A \geq [0] \) be a primitive matrix of order \( n \). Then, there exists, for any \( \varepsilon > 0 \), a natural number \( T_1(\varepsilon) \) such that the vector \( x^i = (A)^j x \) belongs to the \( \varepsilon \)-conic neighborhood \( C_\varepsilon(x^i) \), for all \( x \in \mathbb{R}^n_+ \), and for all \( t > T_1(\varepsilon) \).

**Proof.** We remark that, contrary to the thesis of Theorem 23, here the number \( T_1(\varepsilon) \) does not depend from vector \( x \). Let be \( e^1, e^2, \ldots, e^n \) the \( n \) elementary vectors of \( \mathbb{R}^n \). We associate to each \( e^i \) a number \( T(\varepsilon, e^i) \) such that \( (A)^t e^i \in C_\varepsilon(x_A) \) for all \( t > T(\varepsilon, e^i) \) [Theorem 23]. Let be

\[
T_1(\varepsilon) = \max_{1 \leq i \leq n} T(\varepsilon, e^i).
\]
It holds \((A)^{t}x = \sum_{i=1}^{n} x_i (A)^{t}e_i\), as \(x = \sum_{i=1}^{n} x_i e_i\). Being \(x_i \geq 0\), \(i = 1, ..., n\), and being \(C_e(x_A)\) a convex cone, by Lemma 1, we have \((A)^{t}x \in C_e(x_A)\) for all \(t > T_1(\varepsilon)\).

**Lemma 3.** Let \(A \geq [0]\) be a primitive matrix of order \(n\). Then there exists a number \(T_0\) such that \((A)^{t}x > [0]\) for \(t > T_0\) and for all \(x \geq [0]\).

**Proof.** As \(A\) is irreducible, it holds \(x_A > [0]\). We choose \(\varepsilon_0 > 0\) small enough such that the \(\varepsilon_0\)-conic neighborhood of \(x_A\) contains only positive vectors (except the zero vector). By Lemma 2 we find \(T_0 = T_1(\varepsilon_0)\) such that \((A)^{t}x \in C_{\varepsilon_0}(x_A)\) for all \(t > T_0\), i.e. \((A)^{t}x > [0]\).

**Lemma 4.** There exists a number \(T_0\) such that all solutions \(\{p^t\}\) of problem (9) verify the equalities

\[A'p^t = p^{t+1}\]

for all \(t, 1 \leq t \leq T - T_0\).

**Proof.** We show that vector \(\tilde{x}^T\) of the solution \(\{x^t\}\) of problem (8) is not the zero vector. Indeed, if \(\tilde{x}^T = [0]\), then the maximum of the economic function \(cx^T\) is zero. We consider then a program of problem (8) such that the value of the objective function is greater than zero. We put \(x^t = \alpha(A)^{-1}x_A\), \(t = 1, 2, ..., T\). It is easy to see that with this program all constraints of (8), beginning from the second constraint, become equality relations. If we choose \(\alpha > 0\) small enough, \(x^0 > [0]\) gives \(\alpha \lambda_A^{-1}x_A \leq x^0\) and \(cx^T = \alpha cx_A > 0\). Therefore \(\tilde{x}^T \neq [0]\). We remark that the constraints of the problem imply \(\tilde{x}^t \geq (A)^{-1}\tilde{x}^T\). Taking Lemma 3 into account, we get \(\tilde{x}^t \geq (A)^{T-t}\tilde{x}^T > [0]\) if \(T - t \leq T_0\), i.e. if \(t \leq T - T_0\). By the duality theory it is seen that all constraints associated to the dual problem become equality relations, by substituting the solution.

**Lemma 5.** There exists a number \(T_1\) such that any solution \(\{\tilde{x}^t\}\) of problem (8) satisfies, for all \(t, T_1 \leq t \leq T - T_0\), the relation

\[A\tilde{x}^{t+1} = \tilde{x}^t.\]

**Proof.** From (10) we have

\[p^t = (A')^{t-1}p^1,\quad t = 1, 2, ..., T - T_0.\]

We note that \(p^1 \neq [0]\), as (10) implies \((A')^Tp^1 \geq c \neq [0]\). By applying Lemma 3 to matrix \(A\), we find therefore a number \(T'_0\) such that \(p^t = (A')^{t-1}p^1 > [0]\) for \(t - 1 \geq T'_0\). It remains to put \(T_1 = 1 + T'_0\) and to apply the duality theory.

**Proof of the Turnpike Theorem.** Relation (11) implies \(\tilde{x}^{T-T_0-t} = (A)^t\tilde{x}^{T-T_0}\), therefore we get the thesis of the Turnpike Theorem by applying Lemma 2 to the sequence \(\{\tilde{x}^t\}\), \(T_1 \leq t \leq T - T_0\). By Lemma 2 there exists a number \(T(\varepsilon)\) such that \(\tilde{x}^{T-T_0-t} \in C_e(x_A)\) for all \(t > T(\varepsilon)\). This means that all vectors \(\tilde{x}^t\), with \(t \geq T_1\) and \(t \leq T - T_0 - T(\varepsilon)\) are elements of \(C_e(x_A)\). If we introduce the notation \(T_0' + T(\varepsilon) = T_2(\varepsilon)\), we obtain the thesis of the Turnpike Theorem. We note, moreover, that \(T_1\) is independent from \(\varepsilon\).
5. On a Proposal of Reducibility for a Pair of Matrices in Sraffa Joint-Production Model


If \((A, B)\) are both square of the same order \(n\) (and both nonnegative), a “natural” definition of reducibility of the pair \((A, B)\) is the following one. This pair is reducible if there exist two permutation matrices \(P\) and \(Q\) (with \(Q\) not necessarily coinciding with \(P^T\)) such that

\[
PAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},
\]

with \(A_{11}\) and \(B_{11}\) square of the same order.

There are few economic applications of the said proposal; we point out only Bidard (2004), Moczar (1991b) and Schefold (1971, 1989). For other considerations on irreducibility and reducibility of a pair of nonnegative square matrices, see Giorgi (2007, 2014, 2016) and Giorgi and Magnani (1978).

We have already remarked that a single production Sraffa model, i.e., a Sraffa model where \(B = I\), the basic commodities are those commodities which enter, directly or indirectly, in all production processes. In other words, a Sraffa matrix \(A \geq [0]\), of order \(n\), contains only basic commodities if and only if is irreducible. Otherwise, if \(A\) contains also non-basic commodities, then \(A\) is reducible: see, e.g., Kurz and Salvadori (1995), Pasinetti (1977), Varri (1979), Zaghini (1967). But this definition of basic and non-basic commodities (which holds also for a Leontief single production model) is not suitable for a Sraffa joint production model.

Sraffa (1960) gives, for this last case, a non-formal definition, which may be translated into more formal terms as follows. We recall that \(a_{ij} \geq 0\) denotes the input of commodity \(i\) per unit intensity of process \(j\), \(b_{ij} \geq 0\) denotes the output od commodity \(i\) per unit intensity of process \(j\) \((i, j = 1, 2, \ldots, n)\); we use Leontief’s notation instead of Sraffa’s). Then the pair \((A, B)\) describes, respectively, the input and output matrix of a Sraffa joint production model. It is convenient to assume that both \(A\) and \(B\) have no zero lines. Partition \(A\) and \(B\) conformably

\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

where \(A_1\) is of order \((k,n)\) and \(A_2\) is of order \((n-k,n)\) (the same holds for \(B_1\) and \(B_2\)). According to Sraffa (1960) we have the following definition (see also Bidard and Woods (1989), Schefold (1971, 1989)).
**Definition 4.** The last \((n - k)\) commodities are non-basic according to Sraffa if

\[
rk \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = n - k,
\]

where \(rk[\cdot]\) denotes the rank of the matrix.

Manara (1968) shows that the above definition is equivalent to the following one: there exist two permutation matrices \(P\) and \(Q\) and a matrix \(H\) of order \((n - k, k)\) such that

\[
PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{22}H & A_{22} \end{bmatrix};
\]

\[
PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{22}H & B_{22} \end{bmatrix}
\]

with \(A_{11}\) and \(B_{11}\) square of order \(k\). The commodities referred to the \((n - k)\) rows of \(A_{22}\) are therefore identified as non-basic.

It must be noted that matrix \(H\) solves both the systems:

\[
A_{21} = A_{22}H; \quad B_{21} = B_{22}H.
\]

Then Manara (1968), with reference to a Sraffa pair \((A, B)\), reducible in the sense above specified, introduces the following square matrix of order \(n\):

\[
M = \begin{bmatrix} I_k & [0] \\ -H & I_{n-k} \end{bmatrix}
\]

We have therefore

\[
AM = \begin{bmatrix} A_{11} - A_{12}H & A_{12} \\ [0] & A_{22} \end{bmatrix};
\]

(12)

\[
BM = \begin{bmatrix} B_{11} - B_{12}H & B_{12} \\ [0] & B_{22} \end{bmatrix};
\]

(13)

Manara then uses matrices (12) and (13) to discuss the solutions of the Sraffa system

\[
pB = (1 + r)pA + w\ell,
\]

(14)

where \(p > [0]\) is a price vector, \(r \geq 0\) is the (common) profit rate, \(\ell = [\ell_1, \ell_2, \ldots, \ell_n] > [0]\) is the vector of direct labour requirements per unit intensity of every process and \(w \geq 0\) is the (common) wage rate. Let us partition the price vector \(p\) as \([p^1, p^2]\) and the labour requirements vector \(\ell\) as \([\ell^1, \ell^2]\), with \(p^1\) and \(\ell^1\) having \(k\) elements and \(p^2\) and \(\ell^2\) having \((n - k)\) elements. Let us multiply by \(M\) both members of (14), in order to obtain the equivalent system (matrix \(M\) is non-singular!)

\[
\begin{cases}
p^1(B_{11} - B_{12}H) = (1 + r)p^1(A_{11} - A_{12}H) + w(\ell^1 - \ell^2H) \\
p^1B_{12} + p^2B_{22} = (1 + r)(p^1A_{12} + p^2A_{22}) + w\ell^2,
\end{cases}
\]

(15)
Clearly, the first block of equations can be solved independently of the second block, so that \( p^1 \) contains the prices of the basic commodities, whereas \( p^2 \) “depends” from \( p^1 \) and therefore contains the prices of the non-basic commodities.

On the above results some remarks may be useful.

**Remark 6.** The Sraffa-Manara definition of reducibility of a pair \((A, B), A \geq [0], B \geq [0]\), square of the same order, coincides with the usual notion of reducibility in the case of single production industries, i. e. when \( B = I \). However, if \( B \neq I \), the Sraffa-Manara definition has a quantitative feature, in the sense that it is not independent of the presence of zero elements of \( A \) and \( B \) in certain particular positions, but it depends on the possibility of obtaining every column of the block \( A_{21} \) as a linear combination of the block \( A_{22} \) and simultaneously for the block \( B_{21} \). Therefore it is not a “stable” definition, in the sense that if some elements of \( A_{21}, A_{22}, B_{21}, B_{22} \) vary, it may happen that the matrix \( H \) which solves the systems

\[
A_{21} = A_{22}H; \quad B_{21} = B_{22}H
\]

loses this property.

It must be said that Schefold (1971, 1989), Abraham-Frois and Berrebi (1976), Steedman (1977) and Pasinetti (1980b) have “converted” the Sraffa-Manara definition into the usual (qualitative) definition of reducibility, but not with reference to matrices \((A, B)\). We report the results of Schefold (1971, 1989) who is perhaps the first to point out the said possibility (under suitable assumptions).

**Theorem 25.** Assume \( \det(B) \neq 0, \det(B - A) \neq 0 \). Then the Sraffa pair \((A, B)\) is irreducible in the Sraffa-Manara sense, i. e. it contains only basic commodities, if and only if

1) \( AB^{-1} \) is irreducible; or, equivalently, if and only if
2) \( A(B - A)^{-1} \) is irreducible; or, equivalently, if and only if
3) \( B(B - A)^{-1} \) is irreducible.

Note that in the previous results the matrices \( AB^{-1}, A(B - A)^{-1}, B(B - A)^{-1} \) are not necessarily nonnegative. This fact anticipates, in a sense, the results of the following remark.

**Remark 7.** Let us consider system (14) with \( w = 0 \) and \( r > 0 \) :

\[
\begin{align*}
pB &= \beta pA \\
p &> [0], \quad \beta > 1,
\end{align*}
\]

being \( \beta = (1 + r) \), and the corresponding system involving the “activity vector” of productions \( x \)

\[
\begin{align*}
Bx &= \alpha Ax \\
x &> [0], \quad \alpha > 1.
\end{align*}
\]

It is well-known (recall also what is explained in Section 1 of the present paper and see, e. g., Blakley and Gossling (1967)) that in simple Sraffa production systems, i. e. with \( B = I \),
if $\lambda^*(A)$ is a simple root of the characteristic equation, the presence of only basic commodities (i.e. $A$ is irreducible, in the usual sense), together with the condition of “productivity”, i.e. $\lambda^* < 1$, is both necessary and sufficient to assure the existence of solutions of (16) and (17). Whereas (16) describes the system of “equilibrium prices”, (17) describes a system in quantities which leads to what Sraffa calls “standard commodity vector”. On this concept economists have poured out rivers of ink, sometimes with a complete disregard to formal-mathematical questions. See, e.g., Abraham-Frois and Berrebi (1976), Bellino (2004), Giorgi and Zuccotti (2012), Kurz and Salvadori (1995), Pasinetti (1977), Woods (1978, 1990). The purpose of the present remark is to point out that, contrary to what happens with simple productions (i.e. with $B = I$), if joint productions are assumed ($B \neq I$), then the absence of non-basic commodities is neither necessary nor sufficient to assure the solvability neither of system (16) nor of system (17).

In order to prove the non-necessity of this condition, it is sufficient to choose

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 4/3 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & b_{12} & b_{13} \\ 0 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix},$$

with

$$a_{13} \geq 0, \quad 0 < b_{13} - 2a_{13} \leq 6a_{12},$$
$$b_{12} = 2a_{12} + (2a_{13} - b_{13})/3,$$

in which case $A$ and $B$ have all semipositive lines, the model is productive and profitable, i.e., respectively, it holds

$$Bx > Ax, \quad \text{for some } x > [0]; \quad pB > pA, \quad \text{for some } p > [0],$$

and systems (16) and (17) admit (only) the solutions $(\alpha, x)$ and $(\beta, p)$ described by

$$\alpha = \beta = 2, \quad x_1 > 0, \quad x_2 = 3x_3, \quad x_3 > 0;$$
$$p = (p_1; p_1(b_{13} - 2a_{13}); p_3), \quad \text{with } p_1 > 0, \quad p_3 > 0,$$

although commodities 2 and 3 are non-basic, in the Sraffa-Manara sense, with $P = Q = I$, $k = 1$, $H = [0]$.

The proof that the same condition is not sufficient to make (16) and (17) solvable may be obtained simply by taking into consideration the example given by Manara (1968):

$$A = \begin{bmatrix} 1 & 1.1 \\ 1.1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1.09 & 1.144 \\ 1.144 & 0.99 \end{bmatrix}$$

where both systems (16) and (17) are not solvable, although the positive matrices $A$ and $B$ form a Sraffa model which is productive, profitable and with only basic commodities.

**Remark 8.** Contrary to the von Neumann classical growth model, where matrices $A$ and $B$ are allowed to be of order $(m, n)$, the joint production Sraffa model is square, i.e. $m = n$.  

27
However, the proposal of reducibility given by Manara can be easily fitted to the case of non necessarily square matrices $A$ and $B$. Let us consider the pair $(A, B)$, with $A \geq [0], B \geq [0]$, both of order $(m, n)$. We say that the pair $(A, B)$ is reducible in the Manara sense if and only if there exists a matrix $T$ such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{22}T & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{22}T & B_{22} \end{bmatrix}$$

with $A_{11}$ and $B_{11}$ of order $(m_1, n_1)$ and $T$ of order $(n - n_1, n_1)$.

In this case system (14) may be written in the form

$$\left\{ \begin{array}{l} p^1 B_{11} + p^2 B_{22}T = (1 + r)(p^1 A_{11} + p^2 A_{22}T) + w\ell^1 \\ p^1 B_{12} + p^2 B_{22} = (1 + r)(p^1 A_{12} + p^2 A_{22}) + w\ell^2, \end{array} \right.$$ 

with $p^1 > [0], p^2 > [0], r \geq 0, w \geq 0 (p^1 \in \mathbb{R}^{m_1}, p^2 \in \mathbb{R}^{n-m_1}, \ell^1 \in \mathbb{R}^{n_1}, \ell^2 \in \mathbb{R}^{n-n_1})$. Let us multiply by $T$ the second subsystem:

$$p^1 B_{12}T + p^2 B_{22}T = (1 + r)(p^1 A_{12}T + p^2 A_{22}T) + w\ell^2 T.$$ 

If we subtract this last relation from the first subsystem above, we obtain just the following system

$$p^1 (B_{11} - B_{12}T) = (1 + r)p^1 (A_{11} - A_{12}T) + w(\ell^1 - \ell^2 T),$$

which is the first subsystem of (15), with $T = H$. The second subsystem above coincides, putting $T = H$, with the second subsystem of (15). Therefore we can assert that it is possible to deduce conclusions similar to the square case, even if $m \neq n$ in the pair $(A, B)$.

For other concepts of reducibility (existence of non-basic commodities) in a joint production Sraffa model, see Flaschel (1982). Sanchez Choliz (1992) gives another new definition of basic commodities and basic processes for joint production linear models and applies his definition both to the equilibrium solutions of a von Neumann growth model, and to a Sraffa-like model with joint production.
References


S. CHERUBINO (1956), *Sulle matrici quadrate non negative*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 3a serie, 10, n. 3-4, 217-235.
S. CHERUBINO (1957), *Calcolo delle Matrici*, Edizioni Cremonese, Roma.


J. SANCHEZ CHOLIZ (1992), *Mercancias basicas y subsistema basico en una economia con produccion conjunta*, Revista Española de Economia, 9, 381-399.


H. SCHNEIDER (1977), *The concepts of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov*, Linear Algebra and Its Appl., 18, 139-162.


P. VARRI (1979), *Basic and non-basic commodities in Mr. Sraffa’s price system*, Metroeconomica, 31, 55-72.


