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A Guided Tour in Constraint Qualifications for Nonlinear Programming under Differentiability Assumptions

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A Guided Tour in Constraint Qualifications for Nonlinear Programming under Differentiability Assumptions

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Abstract It is well-known that the celebrated Kuhn-Tucker or Karush-Kuhn-Tucker necessary optimality conditions hold at a local solution point of a nonlinear programming problem if some regularity conditions, usually called "constraint qualifications", are satisfied. In the present paper we give an up-to-date overview of several constraint qualifications proposed in the literature for a nonlinear programming problem, under differentiability assumptions. In particular, we point out the various implications existing among the constraint qualifications considered. For the reader's convenience we shall consider separately the case of inequality constraints only and the case of mixed equality and inequality constraints. Some remarks on second-order constraint qualifications are made and some historical notes on this subject are given.

Key words and phrases Constraint qualifications, nonlinear programming, optimality conditions, Karush-Kuhn-Tucker conditions.

Mathematics Subject Classification (2010): 90C30, 90C46.

1. Introduction

Consider the following classical nonlinear programming problem

$$(\mathcal{P}_1): \qquad \begin{cases} \min f(x) \\ \text{s. t. } g_i(x) \leqq 0, \quad i \in M, \\ h_j(x) = 0, \quad j \in P, \\ x \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$ is an open set, $f : X \longrightarrow \mathbb{R}$, $g_i : X \longrightarrow \mathbb{R}$, $i \in M = \{1, ..., m\}$, $h_j : X \longrightarrow \mathbb{R}$, $j \in P = \{1, ..., p < n\}$. In this paper we assume that f and every g_i are at least differentiable on X, that every h_j is at least continuously differentiable on X and that $(\mathcal{P}_1)$ has at least a local solution $x^0 \in K_1$, where

$$K_1 = \{x \in X : g_i(x) \leqq 0, \quad i \in M, \ h_j(x) = 0, \quad j \in P\}$$

is the *feasible set* of $(\mathcal{P}_1)$.

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It is well-known that if $x^0 \in K_1$ is a local solution of $(\mathcal{P}_1)$ and some *constraint qualification* (henceforth also: "C. Q.") on g_i, $\forall i \in M$, and on h_j, $\forall j \in P$, is satisfied, then the *Kuhn-Tucker conditions* or better, *Karush-Kuhn-Tucker conditions* hold at x^0: there exist multipliers vectors $u = (u_1, ..., u_m) \in \mathbb{R}^m$ and $w = (w_1, ..., w_p) \in \mathbb{R}^p$ such that

$$\text{(KKT):} \qquad \begin{cases} \nabla f(x^0) + \sum_{i=1}^m u_i \nabla g_i(x^0) + \sum_{j=1}^p w_j \nabla h_j(x^0) = 0, \\ u_i g_i(x^0) = 0, \quad i = 1, ..., m, \\ u_i \geqq 0, \quad i = 1, ..., m. \end{cases}$$

The term "constraint qualification" was used for the first time by Kuhn and Tucker (1951) in their pioneering paper on nonlinear programming. Alternatively, also the term "regularity" is used in the literature, especially with reference to "classical" mathematical programming problems, i. e. programming problems with equality constraints only. In our opinion the terms "regularity conditions" should be used only for those conditions involving not only the constraints, but also the objective function. Constraint qualifications also appear in second-order optimality conditions; moreover, they play an important role in deriving duality relations, in the study of sensitivity and stability properties of mathematical programming problems, in the error bound estimates for algebraic systems like systems of equations and/or inequalities, in computational methods, etc.

The first papers involved in constraint qualifications usually take into consideration problem $(\mathcal{P}_1)$, but with only inequality constraints (i. e. $P = \varnothing$), say $(\mathcal{P}_0)$. In this paper we shall consider separately problem $(\mathcal{P}_0)$ and problem $(\mathcal{P}_1)$, in order to obtain a more gradual treatment of the questions concerned with the various C. Q.'s and their relationships.

This paper is organized as follows. Section 2 is concerned with some basic definitions and concepts that will be used in the sequel. Section 3 is concerned with constraint qualifications for a mathematical programming problems with inequality constraints. In section 4 constraint qualifications for problems with both inequality and equality constraints are examined. Section 5 presents some remarks on the so-called "Guignard-Gould-Tolle constraint qualification". Section 6 contains some notes on second-order constraint qualifications; in the final section 7 some historical comments and conclusions are made.

Other overviews of constraint qualifications under differentiability assumptions are contained in Arrow, Hurwicz and Uzawa (1961), Bazaraa, Goode and Shetty (1972), Bazaraa and Shetty (1976), Bazaraa, Sherali and Shetty (2006), Giorgi (1983), Giorgi, Gurraggio and Thierfelder (2004), Mangasarian (1969), Mititelu (1990), Peterson (1973), Solodov (2010), Wang and Fang (2013). Constraint qualifications for nonsmooth optimization problems are treated, e. g., by Castellani (2008), Jourani (1994), Kuntz and Scholtes (1993, 1994), Merkovsky and Ward (1990), Stein (2004), Ward (1991). Quite recently some "sequential" or "asymptotic" or "approximate" optimality conditions which *do not require constraint qualifications at all* have been proposed, both for scalar and for vector optimization problems. See, e. g., Andreani, Martinez and Svaiter (2010), Andreani, Haeser and Martinez (2011), Andreani and others (2015, 2016), Giorgi, Jiménez and Novo (2014, 2016).

There are also necessary optimality conditions and constraint qualifications defined by means of the so-called "image space analysis". See, e. g., Giannessi (2005), Moldovan and Pellegrini (2009 a, b). Finally, there are also several papers concerned with constraint qualifications for vector optimization problems. See, e. g., Giorgi, Jiménez and Novo (2004, 2009), Giorgi and Zuccotti (2011) and the references quoted in the said papers.

2. Preliminaries

In this section we present some definitions and concepts that will be used in the present paper. For any set $S \subset \mathbb{R}^n$ we denote the *closure* of S by $cl(S)$ and denote the set of interior points of S by $int(S)$. $N_\delta(x^0)$ or $U_\delta(x^0)$ is the neighborhood of x^0 of radius $\delta > 0$. If δ is not essential we shall write $N(x^0)$, $U(x^0)$. The index set of *active inequality constraints* at $x^0 \in K_1$ is denoted by $I(x^0)$, i. e.

$$I(x^0) = \left\{ i \in M : g_i(x^0) = 0 \right\}.$$

A set $S \subset \mathbb{R}^n$ is *convex* if $\lambda x + (1-\lambda)y \in S$ for any $x, y \in S$ and every $\lambda \in [0,1]$. The *convex hull* of a set $S \subset \mathbb{R}^n$ is:

$$conv(S) = \left\{ \begin{array}{l} d \in \mathbb{R}^n : d = \sum_{k=1}^r \lambda_k x^k \text{ for some positive integer } r, \\ x^k \in S, \quad \lambda_k \geqq 0 \quad \text{and } \sum_{k=1}^r \lambda_k = 1 \end{array} \right\}.$$

Note that $conv(S)$ is the smallest convex set that contains S.

Let S be a nonempty convex set in $\mathbb{R}^n$. A function $f : S \longrightarrow \mathbb{R}$ is *convex* on S if

$$f(\lambda x + (1-\lambda)y) \leqq \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in S$ and any $\lambda \in [0,1]$.

Let S be a nonempty open convex set in $\mathbb{R}^n$ and let $f : S \longrightarrow \mathbb{R}$ be differentiable on S. Then f is *pseudoconvex* on S if

$$f(y) < f(x) \Longrightarrow \nabla f(x)(y-x) < 0,$$

for any $x, y \in S$, or, equivalently,

$$\nabla f(x)(y-x) \geqq 0 \Longrightarrow f(y) \geqq f(x)$$

for any $x, y \in S$. Let S be a nonempty open convex set in $\mathbb{R}^n$. Then $f : S \longrightarrow \mathbb{R}$ is *quasiconvex on* S if

$$f(\lambda x + (1-\lambda)y) \leqq \max\{f(x), f(y)\}$$

for any $x, y \in S$ and any $\lambda \in [0,1]$. If f is differentiable on the open convex set $S \subset \mathbb{R}^n$, it can be proved (see, e. g., Bazaraa and Shetty (1976), Giorgi, Guerraggio and Thierfelder (2004)) that f is quasiconvex on S if and only if

$$f(y) \leqq f(x) \Longrightarrow \nabla f(x)(y-x) \leqq 0, \quad \forall x, y \in S.$$

Mangasarian (1969) introduced for pseudoconvex functions and for differentiable quasiconvex functions also the *point-wise definitions:* a function $f : S \longrightarrow \mathbb{R}$, S open convex set of $\mathbb{R}^n$, is *pseudoconvex at* $x^0 \in S$ if

$$f(y) < f(x^0) \Longrightarrow \nabla f(x^0)(y - x^0) < 0, \quad \forall y \in S.$$

The function f is *quasiconvex at* $x^0 \in S$ if

$$f(y) \leqq f(x^0) \Longrightarrow \nabla f(x^0)(y - x^0) \leqq 0, \quad \forall y \in S.$$

A function $f : S \longrightarrow \mathbb{R}$ (S convex set of $\mathbb{R}^n$) is *concave (pseudoconcave, quasiconcave)* on S if and only if $-f$ is convex (pseudoconvex, quasiconvex) on S.

A differentiable convex (concave) function is pseudoconvex (pseudoconcave), but the converse is not true. A pseudoconvex (pseudoconcave) function is quasiconvex (quasiconcave), but the converse is not true.

A set $K \subset \mathbb{R}^n$ is a *cone* (with vertex at the origin) if $\lambda x \in K$ for any $x \in K$ and any $\lambda > 0$. A cone $K \subset \mathbb{R}^n$ is a *polyhedral cone* if K can be represented as

$$K = \{x \in \mathbb{R}^n : Ax \leqq 0\},$$

where A is an (m, n) matrix. Hence a polyhedral cone is a closed set.

The *conic hull* of a set $S \subset \mathbb{R}^n$ (or *convex cone generated by* S) is

$$cone(S) = \left\{ \begin{array}{l} d \in \mathbb{R}^n : d = \sum_{k=1}^{r} \lambda_k x^k \text{ for some positive integer } r, \\ x^k \in S, \quad \lambda_k \geqq 0 \end{array} \right\}.$$

Note that $cone(S)$ is the smallest convex cone that contains S and that $0 \in cone(S)$.

Let $S \subset \mathbb{R}^n$ be a nonempty set; then the *(negative) polar or dual cone* of S, denoted by S^*, is given by

$$S^* = \left\{d \in \mathbb{R}^n : d^\top x \leqq 0, \quad \forall x \in S\right\}.$$

If S is empty, then S^* is interpreted as the whole space $\mathbb{R}^n$. Note that S^* is a closed convex cone; moreover, it holds

$$S^* = (cl(S))^*.$$

If $C \subset S$, then it holds $S^* \subset C^*$. One may also define the *bipolar cone* of a set $S \subset \mathbb{R}^n$ by the relation $S^{**} = (S^*)^*$. It results $S \subset S^{**}$, but if S is a nonempty *convex cone* in $\mathbb{R}^n$, it holds

$$S^{**} = cl(S)$$

and if the convex cone S is also closed, then obviously it holds

$$S = S^{**}.$$

More precisely, it can be proved that the above equality holds *if and only if* S is a closed convex cone (see, e. g., Nikaido (1968)). It holds also the relation

$$S^* = cl(conv(S)))^*$$

(S arbitrary set of $\mathbb{R}^n$).

We recall that a cone $C \subset \mathbb{R}^n$ is said to be *finitely generated,* if it is generated by a finite set of vectors in $\mathbb{R}^n$, i. e. if it has the form

$$C = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \mu_i a^i, \quad \mu_i \geqq 0, \ i = 1, ..., m \right\}.$$

It can be proved that the cone C is closed and convex and that a cone is polyhedral if and only if it is finitely generated (*Minkowski-Weyl Theorem*). It results

$$C^* = \left\{ x \in \mathbb{R}^n : a^i x \leqq 0, \ i = 1, ..., m \right\}.$$

We recall also the *Farkas lemma or Farkas theorem of the alternative,* which is an important tool in the study of optimality conditions:

- *Farkas lemma.* Let A be a matrix of order (m, n) and $b \in \mathbb{R}^m$. Then exactly one of the following two systems has a solution (not both):

i) $Ax = b$, $x \in \mathbb{R}^n$, $x \geqq 0$;

ii) $A^\top y \geqq 0$, $b^\top y < 0$, $y \in \mathbb{R}^m$.

See, e. g., Mangasarian (1969).

We now recall the definitions of some *local conic approximations* of a given set $S \subset \mathbb{R}^n$, which are useful to approximate, at least locally at a point $x^0 \in cl(S)$, the set S with a cone, i. e. with a set having a simpler structure. In optimization theory the set S is usually the *feasible set* K_1 or K_0. The development of the theory of mathematical programming since its beginnings was closely connected with conic approximations of sets; for the definitions and basic properties of the various local conic approximations (or tangent cones) used in the literature, see, e. g., Aubin and Frankowska (1990), Bazaraa, Goode and Nashed (1974), Bazaraa and Shetty (1976), Giorgi and Guerraggio (1992, 2002 a,b), Giorgi, Guerraggio and Thierfelder (2004), Palata (1989), Ward (1988).

Definition 1. Let $S \subset \mathbb{R}^n$ and $x^0 \in cl(S)$; the cone

$$T(S, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \exists \left\{ x^k \right\} \subset S, \lim\limits_{k \longrightarrow +\infty} x^k = x^0, \exists \left\{ \lambda_k \right\} \subset \mathbb{R}_+ \\ \text{such that } \ y = \lim\limits_{k \longrightarrow +\infty} \lambda_k (x^k - x^0) \end{array} \right\}$$

is called *Bouligand tangent cone or contingent cone* to S at x^0 (see Aubin and Frankowska (1990), Giorgi and Gurraggio (1992, 2002)).

This cone is closed, but not necessarily convex. The closure of the convex hull of $T(S,x^0)$ is called *pseudotangent cone* to S at x^0 (see Guignard (1969), Gould and Tolle (1972)):

$$P(S,x^0) = cl(conv(T(S,x^0))).$$

Obviously $P(S,x^0)$ is a closed convex cone and it holds $T(S,x^0) \subset P(S,x^0)$.

The Bouligand tangent cone to S at x^0 can be equivalently described in other ways, such as, for example, the following ones:

$$T(S,x^0) = \left\{ y \in \mathbb{R}^n : \exists \{y^n\} \longrightarrow y, \exists t_n \longrightarrow 0, \text{ such that } x^0 + t_n y^n \in S \right\};$$

$$T(S,x^0) = \left\{ y \in \mathbb{R}^n : \exists \{x^n\} \longrightarrow x^0, x^n \in S, \lambda_n \longrightarrow 0 \text{ such that } \frac{x^n - x^0}{\lambda_n} \longrightarrow y \right\};$$

$$T(S,x^0) = \left\{ y \in \mathbb{R}^n : \forall N(y), \forall \lambda > 0, \exists t \in (0,\lambda), \exists \bar{y} \in N(y) \text{ such that } x^0 + t\bar{y} \in S. \right\}$$

Hestenes (1966, 1975) gives the following characterization of the Bouligand tangent cone:

$$T(S,x^0) = \{0 \in \mathbb{R}^n\} \cup \left\{ \begin{array}{l} y \in \mathbb{R}^n : \exists \{x^n\} \subset S, x^n \neq x^0 \text{ for } n \text{ large enough,} \\ x^n \longrightarrow x^0 \text{ and } \frac{x^n - x^0}{\|x^n - x^0\|} \longrightarrow \frac{y}{\|y\|} \end{array} \right\}.$$

Definition 2. Let $S \subset \mathbb{R}^n$ and $x^0 \in cl(S)$; the cone

$$A(S,x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \exists \delta_0 > 0, \exists \text{ a continuous path } \alpha : \mathbb{R} \longrightarrow \mathbb{R}^n \text{ such that} \\ \alpha(\delta) \in S \text{ for } \delta \in (0,\delta_0), \ \alpha(0) = x^0, \text{ and } \lim\limits_{\delta \longrightarrow 0^+} \frac{\alpha(\delta) - \alpha(0)}{\delta} = y \end{array} \right\}$$

is called *cone of attainable directions* to S at x^0 or *Kuhn-Tucker tangent cone* to S at x^0 or also *weak tangent cone* to S at x^0.

It can be proved that $A(S,x^0)$ is a closed cone (see, e. g., Peterson (1973)) and that $A(S,x^0) \subset T(S,x^0)$. Other equivalent expressions of $A(S,x^0)$ are, for example, the following ones:

$$A(S,x^0) = \left\{ y \in \mathbb{R}^n : \forall N(y), \exists \lambda > 0, \forall t \in (0,\lambda), \exists \bar{y} \in N(y) \text{ such that } x^0 + t\bar{y} \in S \right\};$$

$$A(S,x^0) = \left\{ y \in \mathbb{R}^n : \forall \{t_k\} \subset \mathbb{R}_+, t_k \longrightarrow 0^+, \exists \left\{y^k\right\} \longrightarrow y \text{ such that } x^0 + t_k y^k \in S \right\}.$$

Definition 3. Let $S \subset \mathbb{R}^n$ and $x^0 \in cl(S)$; the cone

$$Z(S,x^0) = \left\{ y \in \mathbb{R}^n : \exists \lambda > 0, \forall t \in (0,\lambda] \text{ it holds } x^0 + ty \in S \right\}$$

is called *cone of feasible directions* to S at x^0 or also *radial cone* to S at x^0.

Note that $Z(S,x^0)$ is neither open nor closed; it holds $Z(S,x^0) \subset A(S,x^0)$.

If S is convex, then $Z(S,x^0)$, $A(S,x^0)$ and $T(S,x^0)$ are all convex and it holds

$$cl(Z(S,x^0) = A(S,x^0) = T(S,x^0) = cl(cone(S-x^0)).$$

If S is a convex polyhedron, then $T(S,x^0) = cone(S-x^0)$.

Definition 4. The *normal cone* $N(S,x^0)$ to a convex set $S \subset \mathbb{R}^n$ at $x^0 \in S$ is defined as

$$N(S,x^0) = \left\{ y \in \mathbb{R}^n : (x-x^0)y \leqq 0, \quad \forall x \in S \right\}.$$

When $S \subset \mathbb{R}^n$ is convex, the Bouligand tangent cone and the normal cone at a point $x^0 \in S$ turn out to be polar cones of each other, i. e.

$$T(S,x^0) = \left\{ w \in \mathbb{R}^n : wy \leqq 0, \quad \forall y \in N(S,x^0) \right\},$$

$$N(S,x^0) = \left\{ y \in \mathbb{R}^n : wy \leqq 0, \quad \forall w \in T(S,x^0) \right\}.$$

3. Constraint Qualifications for a Problem with Inequality Constraints

In the present section we condider the problem

$$(\mathcal{P}_0): \quad \begin{cases} \min f(x) \\ \text{s. t. } g_i(x) \leqq 0, \quad \forall i \in M, \\ x \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$ is an open set, $f : X \longrightarrow \mathbb{R}$, $g_i : X \longrightarrow \mathbb{R}$, $i \in M = \{1, ..., m\}$, are at least differentiable on X. We denote by K_0 the feasible set of $(\mathcal{P}_0)$, i. e.

$$K_0 = \{x \in X : g_i(x) \leqq 0, \quad \forall i \in M\}.$$

Definition 5. Let $x^0 \in K_0$; the *linearizing cone or linearized cone or cone of locally constrained directions* at x^0 for $(\mathcal{P}_0)$ is defined as:

$$L(K_0,x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0)y \leqq 0, \quad \forall i \in I(x^0) \right\}.$$

Obviously this cone is closed and convex (being a polyhedral cone). Moreover, $L(K_0,x^0) \neq \varnothing$ because $0 \in L(K_0,x^0)$.

Definition 6. Let $x^0 \in K_0$; the *cone of interior constrained directions* at x^0 for $(\mathcal{P}_0)$ is defined as:

$$L^-(K_0,x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0)y < 0, \quad \forall i \in I(x^0) \right\}.$$

This cone is also referred as the *strictly inward cone* by Bazaraa, Goode and Shetty (1972).

It is easy to see that $L^-(K_0, x^0) = int(L(K_0, x^0))$ and therefore it holds $cl(L^-(K_0, x^0)) = L(K_0, x^0)$ if and only if $L^-(K_0, x^0) \neq \varnothing$.

One of the earliest necessary optimality conditions for $(\mathcal{P}_0)$ was proposed by Fritz John (1948). The *Fritz John (FJ) conditions* for $(\mathcal{P}_0)$ state that if x^0 is a local solution for $(\mathcal{P}_0)$, then there exists a *nonzero* vector $\mu = (\mu_0, \mu_1, ..., \mu_m) \in \mathbb{R}^{m+1}$ such that

$$\text{(FJ):} \qquad \begin{cases} \mu_0 \nabla f(x^0) + \sum_{i=1}^m \mu_i \nabla g_i(x^0) = 0, \\ \mu_i g_i(x^0) = 0, \quad i = 1, ..., m, \\ \mu_i \geqq 0, \quad i = 0, 1, ..., m, \quad \mu \neq 0. \end{cases}$$

In their basic paper of 1951 Kuhn and Tucker (1951) introduced a condition, called "constraint qualification", in order to assure that in (FJ) it holds $\mu_0 \neq 0$, i. e. $\mu_0 > 0$. The celebrated Kuhn-Tucker conditions for $(\mathcal{P}_0)$ or better Karush-Kuhn-Tucker conditions, as also W. Karush (1939) in his master thesis had discovered these conditions, are nothing but the relations (FJ), where all multipliers have been divided by $\mu_0 > 0$ and $u_i = \mu_i / \mu_0$, $i = 1, ..., m$:

$$\text{(KKT):} \qquad \begin{cases} \nabla f(x^0) + \sum_{i=1}^m u_i \nabla g_i(x^0) = 0, \\ u_i g_i(x^0) = 0, \quad i = 1, ..., m, \\ u_i \geqq 0, \quad i = 0, 1, ..., m. \end{cases}$$

We wish to stress that the question of constraint qualifications (for $(\mathcal{P}_0)$ or for the more general problem $(\mathcal{P}_1)$) is *not* related to the *geometric structure* of the feasible set, but instead it is related to the *functional form* of the constraints. For example, let us consider the problem

$$\begin{cases} \min f(x_1, x_2) = x_1 + x_2 \\ \text{s. t. } (-x_1)^3 + x_2 \leqq 0 \\ -x_2 \leqq 0. \end{cases}$$

It is almost immediate to see that the point $x^0 = (0, 0)$ is the optimal solution. However, the Fritz John necessary optimality conditions are satisfied at x^0 only with $\mu_0 = 0$. Indeed, from a geometric point of view, at $x^0 = (0, 0)$ the feasible set presents a *cusp,* which usually invalidates the possibility to obtain the Karush-Kuhn-Tucker conditions. But the true question is that the presence of a cusp is neither necessary nor sufficient to cause the Karush-Kuhn-Tucker conditions to fail at an optimal solution (as a matter of fact, at $x^0 = (0, 0)$ *no constraint qualification* is satisfied for the above problem).

Let us now add to our problem a new constraint:

$$x_2 - 2x_1 \leqq 0.$$

Clearly the feasible set remains the same (the new constraint is therefore *redundant*), but now the Fritz John conditions are satisfied at $x^0 = (0, 0)^\top$ with $\mu_0 > 0$; therefore the Karush-Kuhn-Tucker conditions hold at x^0.

We now take into account the various constraint qualifications proposed in the literature for problem $(\mathcal{P}_0)$, since the pioneering paper of Kuhn and Tucker (1951). First we state two basic results concerning $(\mathcal{P}_0)$: the next Theorems 1 and 2.

Theorem 1. (See, e. g. Bazaraa, Goode and Shetty (1972), Bazaraa and Shetty (1976), Giorgi, Guerraggio and Thierfelder (2004)). Let $x^0 \in K_0$; then it holds

$$L^-(K_0, x^0) \subset Z(K_0, x^0) \subset A(K_0, x^0) \subset T(K_0, x^0) \subset L(K_0, x^0).$$

Proof. First we show that $L^-(K_0, x^0) \subset Z(K_0, x^0)$. Suppose that $y \in L^-(K_0, x^0)$, we need to show there exists $\delta_0 > 0$ such that $x^0 + \delta y \in K_0$, $\forall \delta \in (0, \delta_0]$. Actually, when $i \notin I(x^0)$, we know $g_i(x^0 + \delta y) < 0$ when δ is sufficiently small, because g_i is continuous at x^0. When $i \in I(x^0)$, consider the definition of differentiability of g_i at x^0 :

$$g_i(x^0 + \delta y) = g_i(x^0) + \delta \nabla g_i(x^0) y + o(\delta) \|y\|,$$

for $\delta \longrightarrow 0$. Consequently, $g_i(x^0 + \delta y) < g_i(x^0) = 0$ when δ is sufficiently small. Therefore $y \in Z(K_0, x^0)$ and $L^-(K_0, x^0) \subset Z(K_0, x^0)$.

From the definitions of $Z(K_0, x^0)$, $A(K_0, x^0)$ and $T(K_0, x^0)$ it is easy to check that $Z(K_0, x^0) \subset A(K_0, x^0) \subset T(K_0, x^0)$. Finally we prove that it holds

$$T(K_0, x^0) \subset L(K_0, x^0).$$

Suppose $y \in T(K_0, x^0)$ and there exists $y \in T(K_0, x^0)$ and $i \in I(x^0)$ such that $\nabla g_i(x^0) y > 0$. Then, for $\{x^k\}_{k=1}^{\infty}$ associated with y in the definition of tangent directions, there exists k_0 such that $\nabla g_i(x^0)(x^k - x^0) < 0$ when $k > k_0$. When $k \longrightarrow \infty$, we have

$$g_i(x^k) = g_i(x^0) + \nabla g_i(x^0)(x^k - x^0) + o(\|x^k - x^0\|) =$$

$$\nabla g_i(x^0)(x^k - x^0) + o(\|x^k - x^0\|) > 0.$$

This contradicts the assumption of the feasibility of x^k. Hence $y \in L(K_0, x^0)$ and $T(K_0, x^0) \subset L(K_0, x^0)$. □

Remark 1. By the properties of polar cones we can equivalently write the following inclusion relations:

$$(L(K_0, x^0))^* \subset (T(K_0, x^0))^* \subset (A(K_0, x^0))^* \subset (Z(K_0, x^0))^* \subset (L^-(K_0, x^0))^*.$$

Moreover, it holds also

$$P(K_0, x^0) \subset L(K_0, x^0),$$

being $P(K_0, x^0) = cl(conv(T(K_0, x^0)))$ a closed convex cone and being $L(K_0, x^0)$ a closed convex cone. On the other hand, recall that $(T(K_0, x^0))^* = (P(K_0, x^0))^*$.

Definition 7. Let $x^0 \in K_0$; the *cone of gradients* at x^0 for $(\mathcal{P}_0)$ is defined as

$$C(K_0, x^0) = \left\{ y \in \mathbb{R}^n : y = \sum_{i \in I(x^0)} \lambda_i \nabla g_i(x^0) \text{ for some } \lambda_i \geqq 0,\ i \in I(x^0) \right\}.$$

See Gould and Tolle (1972). The cone of gradients is a closed convex cone and it is easy to prove that $C(K_0, x^0)$ and $L(K_0, x^0)$ are polar cones of each other.

Theorem 2. It holds

$$\begin{aligned} C(K_0, x^0) &= (L(K_0, x^0))^*; \\ L(K_0, x^0) &= (C(K_0, x^0))^*. \end{aligned}$$

Proof. Taking into account that both $L(K_0, x^0)$ and $C(K_0, x^0)$ are closed convex cones, it is enough to prove that $L(K_0, x^0) = (C(K_0, x^0))^*$. Consider any $y \in L(K_0, x^0)$; given $s \in C(K_0, x^0)$ we have

$$ys = \sum_{i \in I(x^0)} \lambda_i y \nabla g_i(x^0).$$

By definition of $L(K_0, x^0)$ and since $\lambda_i \geqq 0$ for all $i \in I(x^0)$, it follows $ys \leqq 0$, so $y \in (C(K_0, x^0))^*$. Conversely, consider $y \in (C(K_0, x^0))^*$, that is $ys \leqq 0$, for all $s \in C(K_0, x^0)$. In particular, since $\nabla g_i(x^0) \in C(K_0, x^0)$, for all $i \in I(x^0)$, we have $y \nabla g_i(x^0) \leqq 0$, completing the proof. □

Consider now a generic constrained minimization problem of the form

$$(\mathcal{P}_A): \qquad \min f(x), \quad x \in A,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable on an open set containing $A \subset \mathbb{R}^n$ and A is *any* subset of $\mathbb{R}^n$: e. g. $A = K_0$ or $A = K_1$ or A is not expressed by a system of inequality and/or equality constraints. In this last case we speak of an *abstract constraint or a set constraint.* With regard to problem $(\mathcal{P}_A)$ the following result, due to Gould and Tolle (1971), Guignard (1969), Hestenes (1966, 1975), Varaiya (1967), is of basic importance.

Theorem 3. If x^0 is a local solution of $(\mathcal{P}_A)$, then the negative of the gradient of f at x^0 lies in the polar cone of the Bouligand tangent cone at x^0, i. e.

$$-\nabla f(x^0) \in (T(A, x^0))^*, \tag{1}$$

that is

$$\nabla f(x^0) y \geqq 0, \quad \forall y \in T(A, x^0),$$

that is

$$\{y : \nabla f(x^0) y < 0\} \cap \{T(A, x^0)\} = \varnothing.$$

Proof. We show that $\nabla f(x^0)y \geqq 0$ for all $y \in T(A, x^0)$. Suppose $y \in T(A, x^0)$. Then, there exist a sequence $\{x^n\} \in A$ converging to x^0 and a nonnegative sequence $\{\lambda_n\} \in \mathbb{R}$ such that $\{\lambda_n(x^n - x^0)\}$ converges to y. Since f is differentiable at x^0, we have, for each n,

$$f(x^n) - f(x^0) = \nabla f(x^0)(x^n - x^0) + o(\|x^n - x^0\|).$$

This implies that

$$\nabla f(x^0)\lambda_n(x^n - x^0) + o(\lambda_n \|x^n - x^0\|) = \lambda_n \left[f(x^n) - f(x^0)\right].$$

Letting $n \longrightarrow \infty$, we have

$$\nabla f(x^0)\lambda_n(x^n - x^0) \longrightarrow \nabla f(x^0)y.$$

Thus $\lambda_n [f(x^n) - f(x^0)]$ has a limit which must be nonnegative since $\lambda_n [f(x^n) - f(x^0)] \geqq 0$ for n sufficiently large. Consequently

$$\nabla f(x^0)y = \lim_{n \longrightarrow \infty} \lambda_n \left[f(x^n) - f(x^0)\right] \geqq 0. \qquad \square$$

Guignard (1969) gives the following formulation of the thesis of Theorem 3:

$$-\nabla f(x^0) \in (P(A, x^0))^*.$$

However, as it is true that for any set $A \subset \mathbb{R}^n$ it holds $A^* = (cl(conv(A)))^*$, we obtain the already recalled equality $(T(A, x^0))^* = (P(A, x^0))^*$. So the condition of Guignard is not sharper than condition (1). If A is *convex,* then we have $T(A, x^0) = cl(cone(A - x^0))$ and in this case (1) becomes

$$-\nabla f(x^0) \in N(A, x^0),$$

where $N(A, x^0)$ is the *normal cone* of A at $x^0 \in A$. This last relation is usually also written as

$$0 \in \nabla f(x^0) + N(A, x^0).$$

To sum up, a basic necessary optimality condition for problem $(\mathcal{P}_0)$ is

$$-\nabla f(x^0) \in (T(K_0, x^0))^*. \tag{2}$$

Condition (2) is generally difficult to apply, as it is not an "algebraic condition", because the contingent cone $T(K_0, x^0)$ is specified in "geometric terms". Therefore, analytically represented optimal conditions would be preferred. Recall that, on the grounds of Theorem 1 it holds $T(K_0, x^0) \subset L(K_0, x^0)$ and therefore $(L(K_0, x^0))^* \subset (T(K_0, x^0))^*$. If it holds

$$(L(K_0, x^0))^* = (T(K_0, x^0))^* \tag{3}$$

then we have

$$-\nabla f(x^0) \in (L(K_0, x^0))^*$$

and, being $(L(K_0, x^0))^* = C(K_0, x^0)$,

$$-\nabla f(x^0) \in C(K_0, x^0).$$

Consequently, there exists $\lambda_i \geqq 0$ for each $i \in I(x^0)$, such that

$$\nabla f(x^0) + \sum_{i \in I(x^0)} \lambda_i \nabla g_i(x^0) = 0.$$

Choosing $\lambda_i = 0$ for $i \notin I(x^0)$, from the previous relations we obtain at once the classical Karush-Kuhn-Tucker conditions for $(\mathcal{P}_0)$:

$$\begin{array}{l} \nabla f(x^0) + \sum_{i=1}^m \lambda_i \nabla g_i(x^0) = 0; \\ \lambda_i g_i(x^0) = 0, \quad i = 1, ..., m; \\ \lambda_i \geqq 0, \quad i = 1, ..., m. \end{array}$$

Summing up, we have obtained the Karush-Kuhn-Tucker conditions for $(\mathcal{P}_0)$ under assumption (3). In other words, any condition on the constraint functions g_i, $i \in I(x^0)$, such that (3) is verified, is considered a *constraint qualification* at $x^0 \in K_0$ for $(\mathcal{P}_0)$. We could also say that constraints qualifications for $(\mathcal{P}_0)$ are conditions that assure that a local conic approximation of the feasible set K_0 at $x^0 \in K_0$, can be recast using some algebraic information on the gradients of the constraints. "In this sense, we can say that a C. Q. is a condition that tries to restrict how the gradients, and hence the constraints, vary *together* in a neighborhood of x^0" (Andreani and others (2012b)).

We shall see that, in one sense, (3) is the weakest constraint qualification for $(\mathcal{P}_0)$. Geometrically, a constraint qualification requires that, for the local approximation of the feasible set K_0 at x^0, the analytic approximation (i. e. the polar of the linearizing cone at x^0) and the geometric approximation (the polar cone of the contingent cone at x^0) be equal.

We now give an overview of the most used constraint qualifications for problem $(\mathcal{P}_0)$.

1. *Guignard-Gould-Tolle* constraint qualification. It is just relation (3):

$$(L(K_0, x^0))^* = (T(K_0, x^0))^*$$

which can be equivalently expressed as

$$L(K_0, x^0) = P(K_0, x^0) = cl(conv(T(K_0, x^0)))$$

or also as

$$L(K_0, x^0) = (T(K_0, x^0))^{**}.$$

See Guignard (1969), Gould and Tolle (1971, 1972). This constraint qualification was also considered by Evans (1970) and by Canon, Cullum and Polak (1970).

2. *Abadie (first) constraint qualification.* It is expressed as:

$$L(K_0, x^0) = T(K_0, x^0).$$

Therefore the Abadie constraint qualification requires that $T(K_0, x^0)$ has to be a (closed) convex cone. It is obvious that this qualification is stronger than the Guignard-Gould-Tolle constraint qualification. Prior to Abadie (1967), this qualification was considered by Hestenes (1966); see also Hestenes (1975).

3. *Karush-Kuhn-Tucker constraint qualification* (see Karush (1939), Kuhn and Tucker (1951)). It is expressed as:

$$L(K_0, x^0) = A(K_0, x^0).$$

This qualification is therefore stronger than the Abadie constraint qualification. We can note that the original definition of the cone $A(K_0, x^0)$ given by Kuhn and Tucker is slightly different from the definition adopted in the present paper, as these authors required that the attainable directions $\alpha(\delta)$ have to be differentiable anywhere in $[0, 1]$. We note, moreover, that the Karush-Kuhn-Tucker constraint qualification is equivalent to the *Hurwicz constraint qualification,* when formulated for a finite-dimensional setting: see Hurwicz (1958) and Arrow, Hurwicz and Uzawa (1961). Finally, we note that Arrow, Hurwicz and Uzawa (1961, page 178, footnote example) take into consideration the closure of the cone $A(K_0, x^0)$, in the erroneous belief that this cone may be not closed. The same mistake is made by Bazaraa, Goode and Shetty (1972) and by Bazaraa and Shetty (1976). A proof that $A(K_0, x^0)$ is closed is given by Peterson (1973).

4. *Arrow-Hurwicz-Uzawa constraint qualification* (see Arrow, Hurwicz and Uzawa (1961)). It is expressed as:

$$L(K_0, x^0) = cl(conv(A(K_0, x^0)))$$

or also, equivalently, as

$$L(K_0, x^0) = (A(K_0, x^0))^{**}.$$

In general, there is no implication between the Abadie constraint qualification and the Arrow-Hurwicz-Uzawa constraint qualification, i. e. between the cones $T(K_0, x^0)$ and $cl(conv(A(K_0, x^0)))$. Abadie (1967) considers the following constraints:

$$\begin{aligned} g_1(x_1, x_2) &= x_2 - (x_1)^2 - s(x_1) \leqq 0, \\ g_2(x_1, x_2) &= -x_2 + (x_1)^2 + c(x_1) \leqq 0, \\ g_3(x_1, x_2) &= (x_1)^2 - 1 \leqq 0, \end{aligned}$$

where

$$s(x_1) = \begin{cases} (x_1)^4 \sin \frac{1}{x_1}, & \text{if } x_1 \neq 0 \\ 0, & \text{if } x_1 = 0, \end{cases}$$

$$c(x_1) = \begin{cases} (x_1)^4 \cos \frac{1}{x_1}, & \text{if } x_1 \neq 0 \\ 0, & \text{if } x_1 = 0. \end{cases}$$

At $x^0 = (0,0)$ it results $A(K_0, x^0) = \{(0,0\}$, whereas $T(K_0, x^0) = \{(x_1, x_2) : x_2 = 0\}$.

On the other hand, Evans (1970) considers the following constraints:

$$\begin{aligned} g_1(x_1, x_2) &= x_1 x_2 \leqq 0, \\ g_2(x_1, x_2) &= -x_1 \leqq 0, \\ g_3(x_1, x_2) &= -x_2 \leqq 0. \end{aligned}$$

At $x^0 = (0,0)$ it results $T(K_0, x^0) = K_0 = \{(x_1, x_2) : x_1 = 0, x_2 \geqq 0\} \cup \{(x_1, x_2) : x_2 = 0, x_1 \geqq 0\}$, whereas $cl(conv(A(K_0, x^0))) = \{(x_1, x_2) : x_1 \geqq 0, x_2 \geqq 0\}$.

Giorgi and Guerraggio (1993, 1998) have proposed a sufficient condition assuring the equivalence between the Abadie constraint qualification and the Kuhn-Tucker constraint qualification.

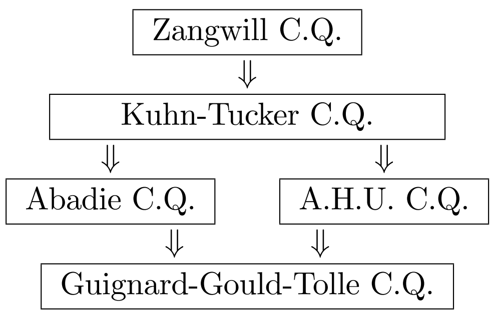
Definition 8. A set $A \subset \mathbb{R}^n$ is said to be *arcwise connected at* $x^0 \in A$ if there exists a continuous vector-valued function (i. e. an arc) $w = w(t) \in A$, $t \in [0,1]$, such that $w(0) = x^0$, $w(1) = x^1$, for any $x^1 \in A$.

Theorem 4. Let $x^0 \in K_0$, with K_0 arcwise connected at x^0 and let the functions $g_i, i \in I(x^0)$, be continuously differentiable at x^0, with $\nabla g_i(x^0) \neq 0$, $\forall i \in I(x^0)$. Then the Abadie constraint qualification is equivalent to the Kuhn-Tucker constraint qualification.

5. *Zangwill constraint qualification* (see Zangwill (1969)). It is expressed as:

$$L(K_0, x^0) = cl(Z(K_0, x^0).$$

We recall that it holds $Z(K_0, x^0) \subset A(K_0, x^0)$. We have therefore the following inclusion diagram concerning the qualifications so far introduced.



For other considerations on constraint qualifications that guarantee, for a convex programming problem $(\mathcal{P}_0)$, the validity of the Zangwill C.Q., the reader is referred to Zhou, Sharifi Mokhtarian and Zlobec (1993).

6. *Cottle constraint qualification* (see Cottle (1963)). It is expressed as:

$$L(K_0, x^0) = cl(L^-(K_0, x^0))$$

or, equivalently, as

$$L^-(K_0, x^0) \neq \varnothing.$$

In its original form this constraint qualification requires that the system

$$\sum_{i \in I(x^0)} u_i \nabla g_i(x^0) = 0, \; u_i \geqq 0, \; i \in I(x^0),$$

has the zero solution only, i. e. the vectors $\nabla g_i(x^0)$, $i \in I(x^0)$, have to be *positively linearly independent.* We recall that the *Gordan theorem of the alternative* (see, e. g., Mangasarian (1969)) states that for any given matrix A, either $Au = 0$, $u \geqq 0$, $u \neq 0$ has a solution u or $yA < 0$ has a solution y, but never both. By applying the Gordan theorem of the alternative, it follows immediately that if the active gradients at x^0 are positively linearly independent,there exists a vector y such that $\nabla g_i(x^0)y < 0$, for all $i \in I(x^0)$, i. e. $L^-(K_0, x^0) \neq \varnothing$ (and viveversa). We note that the Cottle constraint qualification was essentially introduced by Arrow, Hurwicz and Uzawa (1961). It was considered also by Dragomirescu (1967). Bazaraa, Goode and Shetty (1972) express this condition as $L(K_0, x^0) \subset cl(L^-(K_0, x^0))$, noting that the statement $L^-(K_0, x^0) \neq \varnothing$ implies that

$$cl(L^-(K_0, x^0)) \bigcap_{i \in I(x^0)} \left\{ y : \nabla g_i(x^0) y \leqq 0 \right\} = L(K_0, x^0).$$

Indeed, thanks to Theorem 1, if the Cottle C.Q. holds, it will be $L(K_0, x^0) = cl(L^-(K_0, x^0))$.

Now we introduce two variants of the linearizing cone $L(K_0, x^0)$, considered respectively by Abadie (1967) and by Arrow, Hurwicz and Uzawa (1961); this second variant was subsequently generalized by Mangasarian (1969) in the form we take into consideration.

$$L_1(K_0, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \nabla g_i(x^0) y < 0, \text{ if } g_i \text{ is not linear}, \; i \in I(x^0); \\ \nabla g_i(x^0) y \leqq 0, \text{ if } g_i \text{ is linear}, \; i \in I(x^0). \end{array} \right\}$$

$$L_2(K_0, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \nabla g_i(x^0) y < 0, \text{ if } g_i \text{ is not pseudoconcave at } x^0, \; i \in I(x^0); \\ \nabla g_i(x^0) y \leqq 0, \text{ if } g_i \text{ is pseudoconcave at } x^0, \; i \in I(x^0). \end{array} \right\}$$

The reader may note that $g_i(x^0 + \lambda y) \leqq 0$ for all i, provided that $y \in L_2(K_0, x^0)$ and $\lambda > 0$ is sufficiently small.

7. *Abadie constraint qualification II.* It is expressed as:

$$L_1(K_0, x^0) \neq \varnothing.$$

Similarly to the Cottle C.Q. the Abadie C.Q. II can be equivalently expressed as:

$$L(K_0, x^0) \subset cl(L_1(K_0, x^0));$$

see Bazaraa, Goode and Shetty (1972). If the Abadie C.Q. II holds, indeed it will be $L(K_0, x^0) = cl(L_1(K_0, x^0))$.

8. *Arrow-Hurwicz-Uzawa constraint qualification II* (generalized by Mangasarian). It is expressed as:

$$L_2(K_0, x^0) \neq \varnothing.$$

The Arrow-Hurwicz-Uzawa C. Q. II can be equivalently expressed as:

$$L(K_0, x^0) \subset cl(L_2(K_0, x^0)).$$

If the Arrow-Hurwicz-Uzawa C. Q. II holds, indeed it will be $L(K_0, x^0) = cl(L_2(K_0, x^0))$.

This constraint qualification has been originally introduced by Arrow, Hurwicz and Uzawa (1961) who consider concave and non concave constraints. (As already said, the present relaxed form has been proposed by Mangasarian (1969)). We note that if all constraints are *linear (or linear affine),* then the Abadie C.Q. II and the Arrow-Hurwicz-Uzawa C. Q. II are automatically satisfied (a linear (affine) function is both convex and concave), so a *Linear Programming Problem* is automatically qualified. Obviously, if all active constraints at $x^0 \in K_0$ are concave or also pseudoconcave at x^0, then the Arrow-Hurwicz-Uzawa C.Q. II is automatically satisfied. Arrow, Hurwicz and Uzawa (1961) call this case *Reverse constraint qualification.*

Theorem 5. Let $x^0 \in K_0$, then

$$\begin{aligned} L^-(K_0, x^0) &\subset L_1(K_0, x^0) \subset L_2(K_0, x^0) \subset Z(K_0, x^0) \subset \\ &\subset A(K_0, x^0) \subset T(K_0, x^0) \subset L(K_0, x^0). \end{aligned}$$

Proof. By definition $L^-(K_0, x^0) \subset L_1(K_0, x^0) \subset L_2(K_0, x^0)$. We now prove that $L_2(K_0, x^0) \subset Z(K_0, x^0)$. If $y \in L_2(K_0, x^0)$, we claim that $g_i(x^0 + \delta y) \leqq 0$ with $\delta > 0$ being sufficiently small for all $i \in I(x^0)$ such that g_i is pseudoconcave at x^0. Otherwise $g_i(x^0 + \delta y) > 0 = g_i(x^0)$ and then $\delta \nabla g_i(x^0) y > 0$ by the pseudoconcavity. This becomes a contradiction. Hence $y \in Z(K_0, x^0)$. If $y \in L_2(K_0, x^0)$ and $g_i(x)$ is not pseudoconcave at x^0, from $\nabla g_i(x^0) y < 0$ we obtain $g_i(x^0 + \delta y) < 0$ for some $\delta > 0$ sufficiently small (the same is true for $g_i(x)$, with $i \notin I(x^0)$). Hence, in any case $L_2(K_0, x^0) \subset Z(K_0, x^0)$. □

We have therefore the following scheme of inter-relations (recall that the Cottle C.Q. can be equivalently expressed as $L(K_0, x^0) = cl(L^-(K_0, x^0))$; similarly for the Abadie C.Q. II and the A.H.U. C.Q. II).

$$\boxed{\text{Cottle C.Q.}} \implies \boxed{\text{Abadie C.Q.II}} \implies \boxed{\text{A.H.U. C.Q.II}} \implies \boxed{\text{Zangwill C.Q.}}$$

9. *Relaxed Slater constraint qualification.*

This qualification, originally proposed by Slater (1950), has been relaxed by Mangasarian (1969) in the following form:

The constraints g_i, $i \in I(x^0)$, are pseudoconvex at x^0 and there exists $\bar{x} \in K_0$ such that $g_i(\bar{x}) < 0$, $\forall i \in I(x^0)$.

Theorem 6. The relaxed Slater C.Q. implies the Cottle C. Q.

Proof. Suppose that the Slater C.Q. is satisfied. For $i \in I(x^0)$ we have $g_i(\bar{x}) < g_i(x^0) = 0$, therefore we have also $\nabla g_i(x^0)(\bar{x} - x^0) < 0$, because g_i is pseudoinvex at x^0. Let $y = (\bar{x} - x^0)$, then $y \in L^-(K_0, x^0)$. This means $L^-(K_0, x^0) \neq \varnothing$ and hence the Cottle C.Q. holds. □

Remark 2. The original Slater C.Q. is expressed as: the functions g_i, $i = 1, ..., m$, are convex on the convex set $X \subset \mathbb{R}^n$ and there exists a vector $\bar{x} \in K_0$ such that $g_i(\bar{x}) < 0$, $i = 1, ..., m$. This constraint qualification is equivalent to the one considered by Karlin (1959): the functions g_i, $i = 1, ..., m$, are convex on the convex set $X \subset \mathbb{R}^n$ and there is no vector $p \in \mathbb{R}^m$, with $p \geqq 0, p \neq 0$, such that $\sum_{i=1}^m p_i g_i(x) \geqq 0$, $\forall x \in X$. See, e. g., Mangasarian (1969). The original Slater C.Q. and the Karlin C. Q. were proposed in connection with the problem of finding a *Lagrangian saddle point* for $(\mathcal{P}_0)$, a problem that does not require differentiability of the functions involved and that will not be treated in the present paper. Note that under the assumption of convexity and differentiability of the constraints, the original Slater C. Q. is equivalent to the Cottle C.Q.

10. *Linear independence constraint qualification or nondegeneracy constraint qualification* (LI C.Q.)

This constraint qualification is expressed as: the active gradients $\nabla g_i(x^0)$, $i \in I(x^0)$, are linearly independent.

As we have previously remarked that the Cottle C.Q. can be equivalently expressed as: "the active gradients $\nabla g_i(x^0)$, $i \in I(x^0)$, are positively linearly independent", it is obvious that we have the implication

$$\boxed{\text{LI C.Q.}} \quad \Longrightarrow \quad \boxed{\text{Cottle C.Q.}}$$

11. *Strict constraint qualification.*

Let X be a convex set in $\mathbb{R}^n$; the *strict constraint qualification* holds on K_0 if every g_i, $i = 1, ..., m$, is convex on X and K_0 contains at least two distinct points x^1 and x^2 such that every g_i, $i = 1, ..., m$, is strictly convex at x^1 with respect to x^2, i. e.

$$g_i(\lambda x^1 + (1 - \lambda)x^2) < \lambda g_i(x^1) + (1 - \lambda)g_i(x^2), \; i = 1, ..., m, \; \forall \lambda \in (0, 1).$$

This constraint qualification implies the original Slater C.Q. (equivalent to the Karlin C.Q.). Indeed, being $g_i(x^1) \leqq 0$ and $g_i(x^2) \leqq 0$, $i = 1, ..., m$, we have, for $i = 1, ..., m$,

$$g_i(\lambda x^1 + (1 - \lambda)x^2) < \lambda g_i(x^1) + (1 - \lambda)g_i(x^2) \leqq 0, \; \forall \lambda \in (0, 1).$$

Thus the original Slater C.Q. is satisfied:

$$\boxed{\text{Strict C.Q.}} \implies \boxed{\text{Original Slater C.Q.}} \implies \boxed{\text{Relaxed Slater C.Q.}}$$

Remark 3. B. Bernholtz (1964) considers two constraint qualifications: the first one is essentially similar to the classical Kuhn-Tucker C.Q., but the author uses, in his proof, the implicit function theorem instead of alternative or separation theorems, usually utilized in obtaining the necessary optimality conditions for $(\mathcal{P}_0)$. The second one is expressed as follows:
Bernholtz C.Q. II: The index set $I(x^0)$, $x^0 \in K_0$, has cardinality p, with $0 \leqq p \leqq m$; it holds $p \geqq n$ and at x^0 at least one of the Jacobians of order n, that can be constructed from the functions $g_i(x)$, $i \in I(x^0)$, does not vanish.

The above statement cannot be regarded as a constraint qualification: indeed, consider the following example, where the Bernholtz C.Q. II holds at the optimal point $(0, 0)$, but the related Karush-Kuhn-Tucker conditions are not satisfied at that point:

$$\begin{cases} \min(-x_1) \\ \text{s. t. } (x_1)^2 - x_2 \leqq 0, \\ x_2 \leqq 0, \\ -x_1 \leqq 0. \end{cases}$$

For other comments on this question see Giorgi and Guerraggio (1993).

Remark 4. We have already pointed out that the validity of a constraint qualification is in general not related to the presence of cusps in the feasible set or to the shape of the feasible set, but it depends only from the analytical form of the constraint functions. For instance, if we have $g_1(x_1, x_2) = x_2 - (x_1)^3; g_2(x_1, x_2) = -x_2$, the Gould-Tolle-Guignard C.Q. is not satisfied at $x^0 = (0, 0)$, but if the additional and redundant constraint $g_3(x) \leqq 0$ is added, with $g_3(x_1, x_2) = -x_1 - x_2$, the feasible set remains unchanged, but at $x^0 = (0, 0)$ the Gould-Tolle-Guignard C.Q. is now satisfied. An interesting question is therefore the following one: given a general nonlinear programming problem, when it is possible to "regularize" its feasible set by the addition of a finite number of redundant constraints, in order to obtain the validity of a C.Q. at the optimal point, and therefore the validity of the Karush-Kuhn-Tucker conditions? Agunwamba (1977) studies some conditions under which the previous question has a positive answer. Other considerations on similar questions are made by Giorgi (2006).

Remark 5. We have supposed, at the beginning of the present paper, that the set $X \subset \mathbb{R}^n$ is open; Peterson (1973) assumes that the optimal point $x_0 \in K_0$ is interior to $X : x^0 \in int(X)$. Without these assumptions the previous relationships among the various constraint qualifications considered, are not longer all valid, even if it is easy to verify that the implications diagram still holds, even if $x^0 \notin int(X)$, starting from the Zangwill C.Q. (In lemma 6.6.1 of Bazaraa and Shetty (1976) the assumption $x^0 \in int(X)$ is superfluous and actually it is never used in the related proof). For the case of $x^0 \notin int(X)$, the whole implications diagram is preserved if some of the previous constraint qualifications are suitably modified: see Giorgi and Guerraggio (1993, 1994). See also Bertsekas and Ozdaglar (2002), Giorgi (2017) and Giorgi and

Zuccotti (2012) for other considerations on a nonlinear programming problem having (besides inequality and/or equality constraints) also a set constraint (i. e. an abstract constraint, not expressed in a functional form).

Moreover, the constraint qualifications expressed in terms of convexity (concavity) or generalized convexity (generalized concavity) of the constraint functions, can be further relaxed, e. g. by using the notion of *invex functions:* see Giorgi and Guerraggio (1993, 1998). We recall that a differentiable function $f : X \longrightarrow \mathbb{R}$ (X open set of $\mathbb{R}^n$) is called *invex* (see, e. g., Ben-Israel and Mond (1986), Cambini and Martein (2009), Mishra and Giorgi (2008)) if there exists a vector-valued function $\eta(x, u) : X \times X \longrightarrow \mathbb{R}^n$ such that

$$f(x) - f(u) \geqq \nabla f(x, u)\eta(x, u), \quad \forall x, u \in X. \tag{4}$$

Clearly, differentiable convex functions satisfy (4) and it can be proved that pseudoconvex functions are invex, whereas an invex function need not be quasiconvex and a quasiconvex function need not be invex.

4. Constraint Qualifications for a Problem with Inequality and Equality Constraints

In the present section we consider problem $(\mathcal{P}_1)$, i. e.

$$(\mathcal{P}_1): \quad \begin{cases} \min f(x) \\ \text{s. t. } g_i(x) \leqq 0, \ i = 1, ..., m, \\ h_j(x) = 0, \ j = 1, ..., p, \\ x \in X, \end{cases}$$

where X is an open set of $\mathbb{R}^n$, $f : X \longrightarrow \mathbb{R}$ and every $g_i : X \longrightarrow \mathbb{R}$, $i = 1, ..., m$, are differentiable on X and every $h_j : X \longrightarrow \mathbb{R}$, $j = 1, ..., p < n$, is continuously differentiable on X. Weaker differentiability assumptions are possible: see, e. g., Blot (2016), Di (1996), Di and Poliquin (1994), Flett (1980), Giorgi and Zuccotti (2017), Hurwicz and Richter (2003).

We recall that

$$K_1 = \{x \in X : g_i(x) \leqq 0, \ i = 1, ..., m, \ h_j(x) = 0, \ j = 1, ..., p\}$$

is the *feasible set* of $(\mathcal{P}_1)$.

Many definitions and concepts introduced in the previous sections are simply transferred to $(\mathcal{P}_1)$ and to K_1. First we note that it is possible to reformulate $(\mathcal{P}_1)$ by replacing each equality constraint $h_j(x) = 0$ by two inequality constraints $h_j(x) \leqq 0$ and $-h_j(x) \leqq 0$. However, some cautions have to be assumed in this approach: see, e. g., Bazaraa and Shetty (1976). Note that, similarly to what already pointed out for K_0, the cones $T(K_1, x^0)$, $A(K_1, x^0)$ and $Z(K_1, x^0)$ are determined only by the geometric structure of K_1 around the point x^0. The cone of feasible directions $Z(K_1, x^0)$ could be the singleton $\{0\}$, in case $h_j(x)$ is nonlinear for some $j \in P = \{1, ..., p\}$.

The *linearizing cone* at $x^0 \in K_1$ is

$$L(K_1, x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0) y \leqq 0, \ \forall i \in I(x^0), \ \nabla h_j(x^0) y = 0, \ j = 1, ..., p \right\}.$$

In other words,

$$L(K_1, x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0) y \leqq 0, \ \forall i \in I(x^0), \ y \in \ker(\nabla h(x)) \right\}.$$

Similarly to problem $(\mathcal{P}_0)$, it holds, for any $x^0 \in K_1$,

$$T(K_1, x^0) \subset L(K_1, x^0).$$

If we reformulate problem $(\mathcal{P}_1)$ in the equivalent version

$$\begin{cases} \min f(x) \\ \text{s. t. } g_i(x) \leqq 0, \ i = 1, ..., m, \\ h_j(x) \leqq 0, \ j = 1, ..., p, \\ -h_j(x) \leqq 0, \ j = 1, ..., p, \\ x \in X, \end{cases}$$

we see that the cone of interior constrained directions for $(\mathcal{P}_1)$ at x^0, $L^-(K_1, x^0)$ is always the empty set $\varnothing$. Therefore the equality $cl(L^-(K_1, x^0)) = L(K_1, x^0)$ fails to hold. We use therefore the relative interior of $L(K_1, x^0)$ and define the *cone of relative interior constrained directions* at $x^0 \in K_1$:

$$H^-(K_1, x^0) = \left\{y \in \mathbb{R}^n : \nabla g_i(x^0)y < 0,\ \forall i \in I(x^0),\ \nabla h_j(x^0)y = 0,\ j = 1, ..., p\right\}.$$

Indeed, it is easy to check that $H^-(K_1, x^0)$ is the relative interior of $L(K_1, x^0)$, hence $cl(H^-(K_1, x^0)) = L(K_1, x^0)$ if and only if $H^-(K_1, x^0) \neq \varnothing$. In order to obtain inclusion relations for $(\mathcal{P}_1)$ similar to the ones of Theorem 1, we need additional conditions. For example, if the gradients of equality constraints $\{\nabla h_j(x^0), j \in P\}$ are linearly independent, then $H^-(K_1, x^0) \subset A(K_1, x^0) \subset T(K_1, x^0)$. Similarly, if $\{h_j(x), j \in P\}$ are affine and $\{\nabla h_j(x^0), j \in P\}$ are linearly independent, then $H^-(K_1, x^0) \subset Z(K_1, x^0)$.

The following two remarks are fully taken from Wang and Fang (2013).

Remark 6 (Wang and Fang (2013)). Let be

$$\begin{aligned} F_1 &= \{x \in X : g_i(x) \leqq 0,\ i = 1, ..., m\}; \\ F_2 &= \{x \in X : h_j(x) \leqq 0, j = 1, ..., p; -h_j(x) \leqq 0, j = 1, ...p\}. \end{aligned}$$

Denote the cones of tangent directions of F_1 and F_2 at x^0 by $T(F_1, x^0)$ and $T(F_2, x^0)$, respectively, and denote the linearizing cones of F_1 and F_2 at x^0 by $L(F_1, x^0)$ and $L(F_2, x^0)$, respectively. Moreover, denote the cones of interior constrained directions of F_1 and F_2 at x^0 by $L^-(F_1, x^0)$ and $L^-(F_2, x^0)$, respectively. It is easy to verify that $K_1 = F_1 \cap F_2$, $T(K_1, x^0) = T(F_1, x^0) \cap T(F_2, x^0)$, $L(K_1, x^0) = L(F_1, x^0) \cap L(F_2, x^0)$. However, $L^-(K_1, x^0) \neq L^-(F_1, x^0) \cap L^-(F_2, x^0)$, as $L^-(F_2, x^0) = \varnothing$. The cone of relative interior constrained directions is $H^-(K_1, x^0) = L^-(F_1, x^0) \cap L(F_2, x^0)$. The necessary and sufficient condition to have $H^-(K_1, x^0) \subset T(K_1, x^0)$ becomes $L(F_2, x^0) \subset T(F_2, x^0)$, or. equivalently, $L(F_2, x^0) = T(F_2, x^0)$. If the dimension of $T(F_2, x^0)$ is k and there exist $(n-k)$ linearly independent vectors in $\{\nabla h_j(x^0),\ j \in P = \{1, ..., p\}\}$, then $L(F_2, x^0) = T(F_2, x^0)$: see, e. g., Luenberger abd Ye (2008). Note that $n - k \leqq |P|$, where $|P|$ is the cardinality of P. Therefore, the linear independence of $\{\nabla h_j(x^0),\ j \in P\}$ implies $L(F_2, x^0) = T(F_2, x^0)$. Under this condition we have $H^-(K_1, x^0) \subset T(K_1, x^0)$. Similar discussion can be applied to obtain $H^-(K_1, x^0) \subset A(K_1, x^0)$.

Remark 7 (Wang and Fang (2013)). The cone $Z(K_1, x^0)$ may contain no straight line segments if some of the equality constraint functions are nonlinear. In this case $Z(K_1, x^0) = \{0\}$. If $H^-(K_1, x^0) \neq \{0\}$, then the inclusion $H^-(K_1, x^0) \subset Z(K_1, x^0)$ cannot hold. It is easy to verify that if $\{h_j(x), \forall j \in P\}$ are affine and $\{\nabla h_j(x^0), \forall j \in P\}$ are linearly independent, then $H^-(K_1, x^0) \subset Z(K_1, x^0)$.

The following results summarize the main properties and relationships regarding the cones previously considered for problem $(\mathcal{P}_1)$. See also Bazaraa and Shetty (1976), Bazaraa, Sherali and Shetty (2006), Giorgi, Guerraggio and Thierfelder (2004).

Theorem 7. Let $x^0 \in K_1$, then

(i)	$Z(K_1, x^0) \subset A(K_1, x^0) \subset T(K_1, x^0) \subset L(K_1, x^0)$;
(ii)	$cl(H^-(K_1, x^0)) = L(K_1, x^0)$ if and only if $H^-(K_1, x^0) \neq \varnothing$;
(iii)	if $\{\nabla h_j(x^0),\ j = 1, ..., p\}$ are linearly independent, then $H^-(K_1, x^0) \subset A(K_1, x^0) \subset$ $\subset T(K_1, x^0) \subset L(K_1, x^0)$;
(iv)	if $\{h_j(x),\ j = 1, ..., p\}$ are affine and $\{\nabla h_j(x^0),\ j = 1, ..., p\}$ are linearly independent, then $H^-(K_1, x^0) \subset Z(K_1, x^0) \subset A(K_1, x^0) \subset T(K_1, x^0) \subset L(K_1, x^0)$.

The *cone of gradients* at $x^0 \in K_1$ for $(\mathcal{P}_1)$ has the form

$$C(K_1, x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : y = \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0),\ u_i \geqq 0,\ i \in I(x^0), \\ w_j \in \mathbb{R},\ j = 1, ..., p \end{array} \right\}.$$

In the present section we wish, for the reader's convenience, to give the proofs of the properties of this cone (proofs omitted in the previous section, with regard to $C(K_0, x^0)$). For this purpose we need a result which may be considered a variant of the classical *Caratheodory's Theorem.*

Theorem 8 (Caratheodory). Let $y^1, ..., y^r$ be nonzero vectors in $\mathbb{R}^n$, and $x \in \mathbb{R}^n$ such that

$$x = \sum_{i=1}^{r} \gamma_i y^i,$$

with $\gamma_i \geqq 0$ for all $i > m$, with $m < r$. Then, there exist indices subsets $I \subset \{1, ..., m\}$, $J \subset \{m+1, ..., r\}$ and scalars γ_i', $i \in I \cup J$, with $\gamma_i' \geqq 0$ for $i \in J$, such that

$$x = \sum_{i \in I \cup J} \gamma_i' y^i$$

and the vectors y^i, $i \in I \cup J$, are linearly independent.

Proof. If the vectors $y^1, ..., y^r$ are linearly independent, there is nothing to prove. Suppose then that they are linearly dependent. So, there exist scalars α_i with $i = 1, ..., r$, not all zero, such that $\sum_{i=1}^{r} \alpha_i y^i = 0$. Therefore, for all $t \in \mathbb{R}$, $x = \sum_{i=1}^{r} (\gamma_i - t\alpha_i) y^i$. Define $\bar{t}$ as t of minimum absolute value that vanishes one of the coefficients $\gamma_i - t\alpha_i$. Then

$$x = \sum_{i=1}^{r} (\gamma_i - \bar{t}\alpha_i) y^i$$

with $\gamma_i - \bar{t}\alpha_i \geqq 0$ for all $i > m$. Therefore x is written as a linear combination using no more than $r - 1$ vectors. We can repeat this process untill all vectors of the linear combination are linearly independent. □

The next result is a part of the *Minkowski-Weyl Theorem* (see Section 2) and has an autonomous interest.

Theorem 9. For any $x^0 \in K_1$ the cone of gradients $C(K_1, x^0)$ is a closed convex cone (more precisely: a polyhedral cone).

Proof. Consider, without loss of generality, that $I(x^0) = \{1, ..., q\}$. First we prove that $C(K_1, x^0)$ is convex. Consider $s^1, s^2 \in C(K_1, x^0)$ and $t \in [0,1]$. Then, there exist $u, \alpha \in \mathbb{R}^q_+$, $w, \beta \in \mathbb{R}^p$, such that

$$s^1 = \sum_{i=1}^{q} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0)$$

and

$$s^2 = \sum_{i=1}^{q} \alpha_i \nabla g_i(x^0) + \sum_{j=1}^{p} \beta_j \nabla h_j(x^0).$$

Therefore

$$ts^1 + (1-t)s^2 = \sum_{i=1}^{q} (tu_i + (1-t)\alpha_i) \nabla g_i(x^0) + \sum_{j=1}^{p} (tw_j + (1-t)\beta_j) \nabla h_j(x^0).$$

Since $tu_i + (1-t)\alpha_i \geqq 0$ we conclude that $ts^1 + (1-t)s^2 \in C(K_1, x^0)$.

Now we will show that $C(K_1, x^0)$ is closed. For that, consider a sequence $\{s^k\} \subset C(K_1, x^0)$ satisfying $s^k \longrightarrow s^* \in \mathbb{R}^n$. We need to prove that $s^* \in C(K_1, x^0)$. For suitable matrices B and C, we have $C(K_1, x^0) = \{B\zeta + C\lambda, \zeta \geqq 0\}$. By Theorem 8 (Caratheodory), we can assume that $D = (B, C)$ have linearly independent columns, so that $D^\top D$ is nonsingular. Since $\{s^k\} \subset C(K_1, x^0)$, there exists $\gamma^k = \begin{pmatrix} \zeta^k \\ \lambda^k \end{pmatrix}$, with $\zeta^k \geqq 0$, such that

$$s^k = D\gamma^k. \tag{5}$$

Since $D^\top D$ is nonsingular, $\gamma^k = (D^\top D)^{-1} D^\top s^k$. Taking the limit for $k \longrightarrow \infty$, we obtain

$$\begin{pmatrix} \zeta^* \\ \lambda^* \end{pmatrix} = \gamma^* = \lim_{k \longrightarrow \infty} (D^\top D)^{-1} D^\top s^*,$$

with $\zeta^* \geqq 0$. Taking the limit for $k \longrightarrow \infty$ in relation (5), we have $s^* = D\gamma^* \in C(K_1, x^0)$, completing the proof. □

Theorem 10. For any $x^0 \in K_1$, we have $C(K_1, x^0) = (L(K_1, x^0))^*$.

Proof. From the previous theorem, it is enoght to prove that $L(K_1, x^0) = (C(K_1, x^0))^*$. Consider $d \in L(K_1, x^0)$. Given $s \in C(K_1, x^0)$, we have

$$d^\top s = \sum_{i \in I(x^0)} u_i d^\top \nabla g_i(x^0) + \sum_{j=1}^{p} w_j d^\top \nabla h_j(x^0).$$

By definition of $L(K_1, x^0)$ and since $u_i \geqq 0$, it follows that $d^\top s \leqq 0$. So $d \in (C(K_1, x^0))^*$.

Conversely, consider $d \in (C(K_1, x^0))^*$, that is $d^\top s \leqq 0$, for all $s \in C(K_1, x^0)$. In particular, since $\nabla h_j(x^0)$ and $-\nabla h_j(x^0)$ belong to $C(K_1, x^0)$ for all $j = 1, ..., p$, we have $d^\top \nabla h_j(x^0) = 0$. Furthermore, since $\nabla g_i(x^0) \in C(K_1, x^0)$ for all $i \in I(x^0)$, we have $d^\top \nabla g_i(x^0) \leqq 0$, completing the proof. □

Remark 8. A more direct way to obtain the previous result is given by the use of the *Farkas lemma or Farkas theorem of the alternative* (see Section 2), as done by Gould and Tolle (1971). The proof of the Farkas lemma requires, however, the previous proof that a finitely generated cone is closed.

We recall (Theorem 3) that the basic necessary optimality condition for $(\mathcal{P}_1)$ is:

If $x^0 \in K_1$ is a local solution of $(\mathcal{P}_1)$, then

$$\nabla f(x^0) y \geqq 0 \quad \text{for all } y \in T(K_1, x^0),$$

i. e.

$$-\nabla f(x^0) \in (T(K_1, x^0))^*.$$

It turns out that if the equality

$$(T(K_1, x^0))^* = (L(K_1, x^0))^* \tag{6}$$

holds, we can state the classical *Karush-Kuhn-Tucker Theorem* for $(\mathcal{P}_1)$.

Theorem 11 (Karush-Kuhn-Tucker). Let $x^0 \in K_1$ be a local solution for problem $(\mathcal{P}_1)$. If equality (5) holds, then there exist $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ such that

$$\begin{aligned} -\nabla f(x^0) &= \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0), \\ u_i &\geqq 0, \quad i = 1, ..., m, \\ u_i g_i(x^0) &= 0, \quad i = 1, ..., m. \end{aligned}$$

Proof. As we have $-\nabla f(x^0) y \leqq 0$ for all $y \in T(K_1, x^0)$, it holds

$$-\nabla f(x^0) \in (T(K_1, x^0))^* = (L(K_1, x^0))^* = C(K_1, x^0).$$

This means that there exist $u_i \geqq 0$, $i \in I(x^0)$, and $w \in \mathbb{R}^p$ such that

$$-\nabla f(x^0) = \sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0).$$

By choosing $u_i = 0$ for all $i \notin I(x^0)$, we complete the proof. □

We now take into consideration several constraint qualifications proposed (also quite recently) in the literature for problem $(\mathcal{P}_1)$. Let $x^0 \in K_1$.

a) *Guignard-Gould-Tolle C.Q.* It is expressed just by relation (6):

$$(T(K_1, x^0))^* = (L(K_1, x^0))^*,$$

relation that can be equivalently given as

$$P(K_1, x^0) = L(K_1, x^0),$$

where $P(K_1, x^0) = cl(conv(T(K_1, x^0)))$, or also equivalently given as

$$(T(K_1, x^0))^{**} = L(K_1, x^0).$$

We shall revert in Section 5 on this C.Q., as Gould and Tolle (1971) proved that the same is, in one sense, the weakest C.Q.

b) *Abadie C.Q.* It is expressed as:

$$T(K_1, x^0) = L(K_1, x^0).$$

The Abadie C.Q. is called by Bertsekas and Ozdaglar (2002) "Quasiregularity condition". Hestenes (1975) calls "regular" a point $x^0 \in K_1$ which satisfies the Abadie C.Q.

c) *Quasinormality condition.* This condition was proposed by Hestenes (1975). It is expressed as:

- There exist no nonzero vector $(u, w) \in \mathbb{R}^m_+ \times \mathbb{R}^p$ and no sequence $\{x^k\} \longrightarrow x^0$ such that

$$\sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0$$

and for all k, $w_j h_j(x^k) > 0$ for all j, with $w_j \neq 0$, $u_i g_i(x^k) > 0$ for all i with $u_i \neq 0$.

See also Bertsekas and Ozdaglar (2002) and Ozdaglar and Bertsekas (2004). Hestenes (1975) proves that the quasinormality condition implies the Abadie C.Q. Later Bertsekas and Ozdaglar (2002) and Ozdaglar and Bertsekas (2004) modified the quasinormality condition and proposed a "pseudonormality condition" that is stronger than the quasinormality condition. We say that a feasible vector x^0 of $(\mathcal{P}_1)$ is *pseudonormal* if there are no scalars $u_1, ..., u_m, w_1, ..., w_p$ and no sequence $\{x^k\} \subset \mathbb{R}^n$ such that:

(*i*) $\sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0;$

(*ii*) $u_i \geqq 0$ for all $i = 1, ..., m$, and $u_i = 0$ for all $i \notin I(x^0)$; $\sum_{i \in I(x^0)} u_i + \sum_{j=1}^{p} |w_j| > 0;$

(*iii*) $\{x^k\}$ converges to x^0 and $\sum_{i=1}^{m} u_i \nabla g_i(x^k) + \sum_{j=1}^{p} w_j \nabla h_j(x^k) > 0, \ \forall k.$

It was proved by the above quoted authors that pseudormality implies quasinormality and hence also quasiregularity.

d) *Arrow-Hurwicz-Uzawa C.Q.* (Arrow, Hurwicz and Uzawa (1961)). It is expressed as:

$$L(K_1, x^0) = cl(conv(A(K_1, x^0)))$$

or, equivalently as

$$L(K_1, x^0) = (A(K_1, x^0))^{**}.$$

e) *Kuhn-Tucker C.Q.* (Kuhn and Tucker (1951)). It is expressed as:

$$L(K_1, x^0) = A(K_1, x^0).$$

f) *Zangwill C.Q.* (Zangwill (1969)). It is expressed as :

$$L(K_1, x^0) = cl(Z(K_1, x^0)).$$

On the grounds of the previous results we have the following implications diagram.

$$\begin{array}{ccc}
\boxed{\text{Zangwill C.Q.}} & & \\
\Downarrow & & \\
\boxed{\text{Kuhn-Tucker C.Q.}} & \Longrightarrow & \boxed{\text{Arrow-Hurwicz-Uzawa C.Q.}} \\
\Downarrow & & \Downarrow \\
\boxed{\text{Abadie C.Q.}} & \Longrightarrow & \boxed{\text{Guignard-Gould-Tolle C.Q.}} \\
\Uparrow & & \\
\boxed{\text{Quasinormality}} & &
\end{array}$$

It turns out that the Guignard-Gould-Tolle C.Q. is the "weakest" of the constraint qualifications considered in the above diagram. We shall be more precise on this question in the following remark.

Remark 9. The following definition, given by Gould and Tolle (1971), extends the notion of *Lagrange regularity* for $(\mathcal{P}_1)$, given in the pioneering paper of Arrow, Hurwicz and Uzawa (1961).

Definition 9. The triplet (g, h, X) of problem $(\mathcal{P}_1)$, with $X \subset \mathbb{R}^n$, X non necessarily open, is said to be *Lagrange regular* at $x^0 \in K_1$ if and only if for every differentiable objective function f, with a constrained local minimum at x^0, there exist multipliers vectors u and w satisfying the (KKT) conditions:

$$\nabla f(x^0) + \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0,$$

$$u_i g_i(x^0) = 0, \quad i = 1, ..., m;$$
$$u_i \geqq 0, \quad i = 1, ..., m.$$

The following result was proved by Gould and Tolle (1971). See also Bazaraa and Shetty (1976), Bazaraa, Goode, Nashed and Shetty (1976), Giorgi and Zuccotti (2012).

Theorem 12. The triplet (g, h, X) is Lagrange regular for $(\mathcal{P}_1)$ at $x^0 \in K_1$ if and only if the Guignard-Gould-Tolle C.Q. holds at x^0, i. e. if and only if

$$(L(K_1, x^0))^* = (T(K_1, x^0))^*.$$

In their pioneering paper of 1961, Arrow, Hurwicz and Uzawa, considering problem $(\mathcal{P}_0)$, have shown that their C.Q. is the weakest possible for $(\mathcal{P}_0)$, in the sense specified by Definition 9, if the feasible set K_0 is convex. Indeed, in this case, the cones $T(K_0, x^0)$, $A(K_0, x^0)$ and $cl(Z(K_0, x^0))$ collapse to $cl(cone(K_0 - x^0))$. We shall revert further on the present remark in the sequel of this paper.

g) *Mangasarian-Fromovitz C.Q.* (MFC.Q.). It was introduced by Mangasarian and Fromovitz (1967) and is expressed as follows: the gradients $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent and $H^-(K_1, x^0) \neq \varnothing$.

By Theorem 7 (MFC.Q.) is equivalent to: the gradients $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent and $L(K_1, x^0) = cl(H^-(K_1, x^0))$.

We recall (Qi and Wei (2000)) that for $A = \{a^1, ..., a^\ell\}$ and $B = \{b^1, ..., b^s\}$, the pair (A, B) is said to be *positively linearly dependent* if there exist α and β with $\alpha \geqq 0$, $(\alpha, \beta) \neq 0$, such that

$$\sum_{i=1}^{\ell} \alpha_i a^i + \sum_{j=1}^{s} \beta_j b^j = 0.$$

Otherwise, (A, B) is said to be *positively linearly independent.*

(MFC.Q.) is also equivalent to the positive linear independence of the family of gradients $(\{\nabla g_i(x^0), i \in I(x^0)\}, \{\nabla h_j(x^0), j = 1, ..., p\})$. In other words, the relation

$$\sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0$$

has *no nonzero solutions* $u_i \geqq 0$, $w_j \in \mathbb{R}$, $j = 1, ..., p$, for $x^0 \in K_1$. This can be seen by use of the *Motzkin Theorem of the alternative* (see, e. g., Mangasarian (1969)). The (MFC.Q.) is therefore also called *positively linear independence constraint qualification* or also *no nonzero abnormal multiplier constraint qualification* or also *basic constraint qualification.*

If all functions g_i are convex (or even pseudoconvex) and all functions h_j are affine, (MFC.Q.) is equivalent to the well-known *Slater Condition,* we shall consider in the sequel of the present section. (MFC.Q.) has several useful properties, besides of being a constraint qualification for $(\mathcal{P}_1)$ easy to check.

Definition 10. Let $x^0 \in K_1$; the set $\Lambda^g(x^0)$ of *generalized multipliers or Fritz John multipliers* for $(\mathcal{P}_1)$ is defined as the set of *nonzero vectors* $(u_0, u_i, w_j) = (u_0, u_1, ..., u_m, w_1, ..., w_p)$ satisfying the first-order optimality conditions or Fritz John optimality conditions:

$$u_0 \nabla f(x^0) + \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0;$$

$$u_i g_i(x^0) = 0, \quad i = 1, ..., m;$$

$$u_0 \geqq 0, \; u_i \geqq 0, \quad i = 1, ..., m.$$

If a generalized Lagrange multipliers vector (u_0, u, w) is such that $u_0 = 0$, then we say that (u, w) is a *singular* Lagrange multipliers vector. If $u_0 = 1$, then we say that (u, w) is a Karush-Kuhn-Tucker multipliers vector. The set of Karush-Kuhn-Tucker multipliers at x^0 is denoted by $\Lambda(x^0)$.

The following proposition is in part due to Gauvin (1977) and its proof can be found in Bonnans and Shapiro (2000).

Theorem 13. Let x^0 be a local solution of $(\mathcal{P}_1)$. Then the set $\Lambda^g(x^0)$ of generalized Lagrange multipliers is nonempty (i. e. the Fritz John conditions for $(\mathcal{P}_1)$ hold at x^0). Moreover, the following conditions are equivalent:

i) The Mangasarian-Fromovitz C.Q. holds at x^0;

ii) The set of singular Lagrange multipliers is empty;

iii) The set $\Lambda(x^0)$ of Karush-Kuhn-Tucker multipliers is nonempty and bounded; more precisely, $\Lambda(x^0)$ is a compact convex polyhedron.

The (MFC.Q.) is important also for topological stability of the feasible set K_1; see Guddat and others (1986). Günzel and Jongen (2006) have proved that the so-called "strong stability" for $(\mathcal{P}_1)$ implies the (MFC.Q.). Robinson (1976 a) has shown the equivalence of the (MFC.Q.) and a form of local stability of the set of solutions of a system of inequalities. In addition, Gauvin and Tolle (1977) established that the (MFC.Q.) is preserved under right-hand side perturbations of $(\mathcal{P}_1)$. However, if some equality $h_j(x) = 0$ is written as a double inequality, i. e. $h_j(x) \leqq 0$, $-h_j(x) \leqq 0$, this constraint qualification cannot be satisfied. This drawback of the (MFC.Q.), as well as of the (LIC.Q.), does not affect the Constant Rank Constraint Qualification (CRC.Q.) we shall introduce in the sequel.

Other interesting results concerning (MFC.Q.) are contained in the paper of Still and Streng (1996). Finally, we point out (see, e. g., Bonnans (2006)) that if in $(\mathcal{P}_1)$ the functions f and every g_i, $i = 1, ..., m$, are *convex* and every h_j, $j = 1, ..., p$, is *linear,* the set $\Lambda(x)$, if nonempty, is the *same* for every x solution of $(\mathcal{P}_1)$. In other words, in this case $\Lambda(x)$ is independent from each vector x solution of the said problem.

Kyparisis (1985) has introduced a stronger version of (MFC.Q.), called *Strict Mangasarian-Fromovitz C.Q.* (SMFC.Q.). We say that (SMFC.Q.) is satisfied at $x^0 \in K_1$, with Karush-Kuhn-Tucker multipliers vectors (u, w), if:

i) the vectors $\{\nabla g_i(x^0),\ i \in I^+(x^0)\} \cup \{\nabla h_j(x^0),\ j = 1, ..., p\}$ are linearly independent;
ii) there exists $d \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(x^0)d &< 0, \quad i \in I(x^0) \diagdown I^+(x^0); \\ \nabla g_i(x^0)d &= 0, \quad i \in I^+(x^0); \\ \nabla h_j(x^0)d &= 0, \quad j = 1, ..., p. \end{aligned}$$

Here $I^+(x^0)$ denotes the set of *strictly active inequality constraints* in $(\mathcal{P}_1)$, i. e.

$$I^+(x^0) = \left\{ i : i \in I(x^0),\ u_i > 0 \right\}.$$

Note that $I^+(x^0)$ depends on the multiplier vector u. Kyparisis (1985) shows that (SMFC.Q.) is both necessary and sufficient to have uniqueness of Karush-Kuhn-Tucker multipliers, i. e. for $\Lambda(x^0)$ to be a singleton. Moreover, (SMFC.Q.) has other useful properties in the study of second-order optimality conditions for $(\mathcal{P}_1)$ and in the study of stability and sensitivity analysis of nonlinear parametric programming problems. We have however to note that (SMFC.Q.) depends on the a priori knowledge of the sign of the Karush-Kuhn-Tucker multipliers vectors, which depend (indirectly) also from the objective function. Strictly speaking, the (SMFC.Q.) cannot be considered a "proper" C.Q., but only a "regularity condition" (see also Wachsmuth (2013)).

As already remarked, the Mangasarian-Fromovitz C.Q. implies the Zangwill C.Q. under the condition that every h_j, $j = 1, ..., p$, is affine. Otherwise we have (Theorem 7):

$$\boxed{\text{MFC.Q.}} \Longrightarrow \boxed{\text{KTC.Q.}} \Longrightarrow \boxed{\text{Abadie C.Q.}}$$

See also Bazaraa and Shetty (1976) and Bazaraa, Sherali and Shetty (2006). For the reader's convenience we give an authonomous proof of the said implications.

Theorem 15. If (MFC.Q.) holds at $x^0 \in K_1$, then

$$A(K_1, x^0) = T(K_1, x^0) = L(K_1, x^0).$$

Proof. We recall first that it holds, for any $x^0 \in K_1$,

$$T(K_1, x^0) \subset L(K_1, x^0).$$

Let us suppose that (MFC.Q.) is satisfied at $x^0 \in K_1$. We have to prove that

$$L(K_1, x^0) \subset A(K_1, x^0) \subset T(K_1, x^0).$$

Hence, let $y \in L(K_1, x^0)$ and such that

$$\nabla g_i(x^0)y < 0,\ \forall i \in I(x^0);\ \nabla h_j(x^0)y = 0,\ j = 1, ..., p.$$

We claim that $y \in A(K_1, x^0)$. Define the application $H : \mathbb{R}^{p+1} \longrightarrow \mathbb{R}^p$ as follows

$$H_j(d,t) = h_j(x^0 + ty + (\nabla h(x^0))^\top d), \quad j = 1, ..., p$$

where $\nabla h(x^0)$ denotes the Jacobian of h at x^0. The nonlinear equation $H(d,t) = 0$ has the solution $(\bar{d}, \bar{t}) = (0,0)$ with

$$\nabla_d H(0,0) = \nabla h(x^0)(\nabla h(x^0))^\top$$

and the latter matrix is nonsingular (even positive definite) due to the linear independence of the vectors $\nabla h_j(x^0)$, $j = 1, ..., p$. The *Implicit Function Theorem* yields a $\mathcal{C}^1$-function $d : (-\epsilon, \varepsilon) \longrightarrow \mathbb{R}^p$ such that $d(0) = 0$, $H(d(t), t) = 0$ and $d'(t) = -(\nabla_d H(d(t), t)^{-1}(\nabla_t H(d(t), t)$ for all $t \in (-\varepsilon, \varepsilon)$. Hence we have

$$\begin{aligned} d\,'(0) &= -(\nabla_d H(0,0))^{-1} \nabla_t H(0,0) = \\ &= -(\nabla_d (H(0,0)^{-1} \nabla h(x^0) y = 0. \end{aligned}$$

Now put $x(t) = x^0 + ty + (\nabla h(x^0))^\top d(t)$ for all $t \in (-\varepsilon, \varepsilon)$. Reducing ε if necessary, we have proved that $x : (-\epsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$-curve, with $x(0) = x^0$, $x'(0) = y$ and $h_j(x(t)) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Moreover, by continuity we have $g_i(x(t)) < 0$ for all $i \notin I(x^0)$ and t sufficiently small. For $i \in I(x^0)$ we have $g_i(x(0)) = g_i(x^0) = 0$ and

$$\frac{d}{dt} g_i(x(0)) = \nabla g_i(x^0) y < 0$$

and hence $g_i(x(t)) < 0$ for all t sufficiently small. Hence $x(t) \in K_1$ for all $t \in [0, \varepsilon)$, $x(0) = x^0$ and $x'(0) = y$. So, $y \in L(K_1, x^0)$ by assumption is also an element of $A(K_1, x^0)$. Being $A(K_1, x^0) \subset T(K_1, x^0)$, this shows that $L(K_1, x^0) \subset A(K_1, x^0) \subset T(K_1, x^0)$. Being $T(K_1, x^0) \subset L(K_1, x^0)$, it results that all these cones are equal and hence the thesis is proved. □

Theorem 15 is proved also by Forst and Hoffmann (2010), by Hiriart-Urruty (2007) and by Still and Streng ((1996), Lemma 2.1). The proof of Bazaraa and Shetty (1976) and Bazaraa, Sherali and Shetty (2006) is a bit more intricate.

Similarly to what done in the previous section with reference to $(\mathcal{P}_0)$, we can relax the Mangasarian-Fromovitz C.Q. by introducing the second Abadie C.Q. and the second Arrow-Hurwicz-Uzawa C.Q.

h) *Abadie C.Q. II.* (Abadie (1967)). It is expressed as: the vectors $\nabla h_j(x0)$, $j = 1, ..., p$, are linearly independent and $H_1^-(K_1, x^0) \neq \varnothing$, where

$$H_1^-(K_1, x^0) = \left\{ \begin{array}{r} y \in \mathbb{R}^n : \nabla g_i(x^0) y < 0, \ i \in I(x^0), \ g_i \text{ is nonlinear}, \\ \nabla g_i(x^0) y \leqq 0, \ i \in I(x^0), \quad g_i \text{ is linear}, \\ \nabla h_j(x^0) = 0, \qquad j = 1, ..., p. \end{array} \right\}$$

i) *Arrow-Hurwicz-Uzawa C.Q.II.* (Arrow, Hurwicz and Uzawa (1961), Mangasarian (1969)). It is expressed as: he vectors $\nabla h_j(x0)$, $j = 1, ..., p$, are linearly independent and $H_2^-(K_1, x^0) \neq \varnothing$, where

$$H_2^-(K_1, x^0) = \left\{ \begin{array}{c} y \in \mathbb{R}^n : \nabla g_i(x^0)y < 0, \ i \in I(x^0), \ g_i \text{ is non pseudoconcave at } x^0, \\ \nabla g_i(x^0)y \leqq 0, \ i \in I(x^0), \ \ g_i \text{ is pseudoconcave at } x^0, \\ \nabla h_j(x^0) = 0, \qquad\qquad j = 1, ..., p. \end{array} \right\}$$

If every h_j, $j = 1, ..., p$, is affine we have the implications

$$\boxed{\text{MFC.Q.}} \Longrightarrow \boxed{\text{AbadieC.Q.}} \Longrightarrow \boxed{\text{AHUC.Q.II}} \Longrightarrow \boxed{\text{Zangwill C.Q.}}$$

Otherwise we have the implications

$$\boxed{\text{MFC.Q.}} \Longrightarrow \boxed{\text{Abadie C.Q.}} \Longrightarrow \boxed{\text{AHUC.Q.II}} \Longrightarrow \boxed{\text{Kuhn-Tucker C.Q.}}$$

If all g_i, $i \in I(x^0)$, are pseudoconcave at $x^0 \in K_1$ and the vectors $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent, the second Arrow-Hurwicz-Uzawa C.Q. holds automatically. Following these authors, we may call this case "reverse constraint qualification".

j) *Linear Independence C.Q.* (LIC.Q.). It is expressed as:

the gradients $\{\nabla g_i(x^0), i \in I(x^0); \nabla h_j(x^0), j = 1, ..., p\}$ are linearly independent. Obviously we have the implication

$$\boxed{\text{LIC.Q.}} \Longrightarrow \boxed{\text{MFC.Q.}}$$

k) *Slater C.Q.* (Mangasarian (1969)). In its extended form it is expressed as: g_i is pseudoconvex at x^0, for every $i \in I(x^0)$, and every h_j, $j = 1, ..., p$, is *quasilinear at* x^0, i. e. is both quasiconvex and quasiconcave at x^0; the gradients $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent and there exists $\bar{x} \in K_1$ such that $g_i(\bar{x}) < 0$, $i \in I(x^0)$, and $h_j(\bar{x}) = 0$, $j = 1, ..., p$.

Theorem 16. The Slater C.Q. implies the MFC.Q.:

$$\boxed{\text{Slater C.Q.}} \Longrightarrow \boxed{\text{MFC.Q.}}$$

Proof. Let $y = \bar{x} - x^0$. We only need to prove that it holds $\nabla h_j(x^0)y = 0$ for all $j = 1, ..., p$. For each $h_j(x)$ and all $\lambda \in [0, 1]$, we have $h_j(\lambda\bar{x} + (1-\lambda)x^0) \leqq \max\{h_j(x^0), h_j(\bar{x})\} = 0$ because $h_j(x)$ is quasiconvex at x^0. Note that $h_j(x)$ is also quasiconcave at x^0. Hence $h_j(\lambda\bar{x}+(1-\lambda)x^0) \geqq \min\{h_j(x^0), h_j(\bar{x})\} = 0$. Consequently, $h_j(\lambda\bar{x} + (1-\lambda)x^0) = 0$. By the Taylor series expansion of $h_j(\lambda\bar{x} + (1-\lambda)x^0)$ at x^0, we have

$$\lambda\nabla h_j(x^0)y + o(\lambda)\,\|y\| = 0,$$

as $\lambda \longrightarrow 0$. Similarly to the proof of Theorem 6, we can prove that $\nabla g_i(x^0)y < 0$, $\forall i \in I(x^0)$. Therefore $y \in H^-(K_1, x^0)$ and $H^-(K_1, x^0) \neq \varnothing$. The Mangasarian-Fromovitz C.Q. is then satisfied. $\square$

Note that if in $(\mathcal{P}_1)$ the constraints $h_j(x)$, $j = 1, ..., p$, are linear affine, with gradient a^j, in case $\{a^1, a^2, ..., a^p\}$ is a linearly dependent set, then we can always choose a linearly independent subset of it, say $\{a^1, ..., a^k\}$ such that $span\,\{a^1, ..., a^k\} = span\,\{a^1, ..., a^p\}$. Moreover, keeping only the constraints h_j, $j = 1, ..., k$, in the formulation of $(\mathcal{P}_1)$, does not change its feasible region. That's why some authors formulate the extended Slater C.Q. for $(\mathcal{P}_1)$ assuming the functions h_j, $j = 1, ..., p$, linear affine, without mentioning the linear independence of their gradients (see, e. g., Bazaraa and Shetty (1976), Solodov (2010)).

l) *Constant Rank Constraint Qualification* (CRC.Q.).

It was introduced by Janin (1984) and is expressed as follows: there exists a neighborhood $U(x^0)$ of $x^0 \in K_1$ such that for every pair of subsets $I_1(x^0) \subset I(x^0)$ and $J_1 \subset \{1, ..., p\}$, the set of gradients $\{\nabla g_i(x), \nabla h_j(x),\ i \in I_1(x^0), j \in J_1\}$ has the same rank for all $x \in U(x^0) \cap K_1$.

It appears that the rank in question depends on the choice of $I_1(x^0)$ and J_1 but not on the point $x \in U(x^0) \cap K_1$. Clearly, (LIC.Q.) implies (CRC.Q.) and (CRC.Q.) can be viewed as a relaxation of (LIC.Q.). Linearity of constraints also implies (CRC.Q.). The condition (CRC.Q.) is indeed a constraint qualification: Janin (1984) proved that (CRC.Q.) implies the Abadie C.Q. However, (CRC.Q.) is neither weaker nor stronger than (MFC.Q.). In other words

$$(MFC.Q.) \nRightarrow (CRC.Q.)$$

and

$$(CRC.Q.) \nRightarrow (MFC.Q.).$$

Obviously (CRC.Q.)$\nRightarrow$(SMFC.Q.) and also (SMFC.Q.)$\nRightarrow$(CRC.Q.). See Liu (1995).

Note also that, unlike (MFC.Q.), if (CRC.Q.) holds at $x^0 \in K_1$, it will continue to hold if any of the equality constraints $h_j(x) = 0$ were to be replaced by two inequalities $h_j(x) \leqq 0$, $-h_j(x) \leqq 0$. Lu(2011) proved that (MFC.Q,) and (CRC.Q.) are related in the following sense: if (CRC.Q.) holds, there exists an alternative representation of the feasible set K_1 for which (MFC.Q.) holds. Moreover, (CRC.Q.) implies the condition (CPLD): see further.

m) *Relaxed Constant Rank Constraint Qualification* (RCRC.Q.).

It was introduced by Minchenko and Stakhovski (2011a) and is expressed as follows: there exists a neighborhood $U(x^0)$ of $x^0 \in K_1$ such that for every index set $I_1(x^0) \subset I(x^0)$ the set

$$\left\{\nabla g_i(x),\ i \in I_1(x^0)\right\} \cup \{\nabla h_j(x),\ j = 1, ..., p\}$$

has the same rank for all $x \in U(x^0) \cap K_1$.

See also Minchenko and Stakhovski (2011b). It is clear that (CRC.Q.) implies (RCRC.Q.). It can be seen from the following example, taken from Minchenko and Stakhovski (2011a), that the reverse implication is not valid:

$K_1 = \{x \in \mathbb{R}^2 : x_1 - x_2 = 0,\ -x_1 \leqq 0,\ -x_1 - (x_2)^2 \leqq 0\}$, $x^0 = (0, 0)$.

The condition (RCRC.Q.) is indeed a constraint qualification, as it implies the Abadie C.Q.. When there are no inequality constraints, (RCRC.Q.) reduces to the *weak constant rank* *condition* introduced by Penot (1986). The condition (RCRC.Q.) has been applied to the study of a parametric constraint system by Lu (2012).

n) *Constant Positive Linear Dependence Condition* (CPLD). It was introduced by Qi and Wei (2000) and is expressed as follows: there exixts a neighborhood $U(x^0)$ of $x^0 \in K_1$ such that whenever for some index set $I_1(x^0) \subset I(x^0)$ and $J_1 \subset \{1, ..., p\}$, the system

$$\sum_{i \in I_1(x^0)} u_i \nabla g_i(x^0) + \sum_{j \in J_1} w_j \nabla h_j(x^0) = 0, \quad u_i \geqq 0, \ \forall i \in I_1(x^0),$$

has a nonzero solution, the set

$$\left\{ \nabla g_i(x), \ i \in I_1(x^0) \right\} \cup \{ \nabla h_j(x), \ j \in J_1 \}$$

is linearly dependent for all $x \in U(x^0)$.

Comparing the dual form of (MFC.Q.) with (CPLD), it is immediate that (MFC.Q.) implies (CPLD), but not vice versa. Andreani and others (2005) have proved that (CPLD) implies the quasinormality condition, and hence the Abadie C.Q.. Moreover, (CRC.Q.) implies (CPLD). Finally, (CPLD) is neither weaker nor stronger than (RCRC.Q.), as it can be seen from the following example, taken from Minchenko and Stakhovski (2011a):

$K_1 = \{x \in \mathbb{R}^2 : x_2 = 0, \ x_1 - (x_2)^2 \leqq 0, \ -x_1 - (x_2)^2 \leqq 0\}$, $x^0 = (0, 0)$.

It is easy to check that the conditions (CRC.Q.), (MFC.Q.) and (CPLD) do not hold at $x^0 \in K_1$, whereas K_1 satisfies at $x^0 = (0, 0)$ the condition (RCRC.Q.).

o) *Relaxed Constant Positive Linear Dependence Condition* (RCPLD). It was proposed by Andreani and others (2012a) and is expressed as follows: for any subset $I_1(x^0) \subset I(x^0)$ and some $J_1 \subset \{1, ..., p\}$ such that $\{\nabla h_j(x^0), \ j \in J_1\}$ form a basis of $\{\nabla h_j(x^0), \ j = 1, ..., p\}$, if there exists a nonzwero solution of the linear system

$$\sum_{i \in I_1(x^0)} u_i \nabla g_i(x^0) + \sum_{j \in J_1} w_j \nabla h_j(x^0) = 0, \quad u_i \geqq 0, \ \forall i \in I_1(x^0),$$

then there exists a neighborhood $U(x^0)$ such that the gradients

$$\left\{ \nabla g_i(x), \nabla h_j(x), \ i \in I(x^0), \ j \in J_1 \right\}$$

are linearly dependent for all $x \in U(x^0)$ and $\{\nabla h_j(x), \ j = 1, ..., p\}$ has the same rank for all $x \in U(x^0)$.

Note that (CPLD) imposes conditions on all subsets of $\{1, ..., p\}$ while (RCPLD) only needs a certain one of them. It was proved by Andreani and others (2012a) that (CPLD) implies (RCPLD) and that this last condition implies the Abadie C.Q. and hence it is a true constraint qualification. Summing up: the (CRC.Q.), the (RCRC.Q.) and the Weak Constant Rank (WCR) condition (that will be introduced in Section 6) may be considered as relaxations of the (LIC.Q.). The (CPLD) and the (RCPLD) conditions may be considered as relaxations of the (MFC.Q.).

On the grounds of what previously stated, we have the following schemes of implications. See also Andreani, Martinez and Schuverdt (2005) and Andreani and others (2012b).

$$\begin{array}{ccccc}
\boxed{\text{pseudonormality}} & \Longrightarrow & \boxed{\text{quasinormality}} & \Longrightarrow & \boxed{\text{Abadie C.Q.}} \\
 & & \Uparrow & & \Uparrow \\
 & & \boxed{\text{CPLD}} & \Longrightarrow & \boxed{\text{RCPLD}} \\
 & & \Uparrow \quad\quad \Uparrow & & \\
 & & \boxed{\text{CRC.Q}} \quad \boxed{\text{MFC.Q.}} & \Longleftarrow & \boxed{\text{Slater C.Q.}} \\
 & & \Uparrow \quad\quad \Uparrow & & \\
 & & \boxed{\text{LIC.Q.}} & &
\end{array}$$

$$\begin{array}{ccccc}
\boxed{\text{CRC.Q}} & \Longrightarrow & \boxed{\text{RCRC.Q.}} & \not\Leftrightarrow & \boxed{\text{CPLD}} \\
 & & \Downarrow & & \\
 & & \boxed{\text{RCPLD}} & \not\Rightarrow & \boxed{\text{quasinormality}} \\
 & & \Downarrow & & \\
 & & \boxed{\text{Abadie C.Q.}} & &
\end{array}$$

$$\begin{array}{ccc}
\boxed{\text{MFC.Q.}} & \Longrightarrow & \boxed{\text{pseudonormality}} \\
\Downarrow & & \Downarrow \\
\boxed{\text{CPLD}} & \Longrightarrow & \boxed{\text{quasinormality}}
\end{array}
\qquad\qquad
\boxed{\text{CPLD}} \not\Leftrightarrow \boxed{\text{pseudonormality}}$$

Other considerations on the last types of constraint qualifications examined and on their extensions are contained in the papers of Andreani and others (2012b), Andreani and Silva (2014), in the paper of Kruger, Minchenko and Outrata (2014), in the paper of Guo, Zhang and Lin (2014) and in the paper of Penot (1986). Unfortunately the Penot constraint qualification is more difficult to verify than the other C.Q.'s considered by the authors quoted above.

Remark 10. A more general format of constraints is given by

$$K_2 = \{x \in \mathbb{R}^n : S(x) \in Q\},$$

where Q is a subset of $\mathbb{R}^s$ and $S : \mathbb{R}^n \longrightarrow \mathbb{R}^s$. Clearly, K_0 and K_1 are special cases of K_2. When Q is a *closed convex set,* the *Robinson C.Q.* holds at $x^0 \in K_2$ if

$$0 \in int \left\{ S(x^0) + \operatorname{Im} \nabla S(x^0) - Q \right\}.$$

See Robinson (1976a), Bonnans and Shapiro (2000), Zowe and Kurcyusz (1979). Robinson C.Q. ensures that

$$T(K_2, x^0) = \left\{ y \in \mathbb{R}^n : \nabla S(x^0) y \in T(Q, S(x^0)) \right\}.$$

Furthermore, if Q is a *closed convex cone* and x^0 is a local solution of the related constrained minimization problem, then there exists $v \in \mathbb{R}^s$ such that

$$\nabla f(x^0) + (\nabla S(x^0))^\top v = 0,$$

$$S(x^0) \in Q,\ v \in Q^*,\ vS(x^0) = 0.$$

In the case of a finite number of equality and inequality constraints, i. e. $K_2 = K_1$, the Robinson C.Q. reduces to the (MFC.Q.).

Remark 11. An important property of parametric constraint systems is the *metric regularity* (see, e. g., Borwein (1986), Cominetti (1990), Li (1997), Lu (2012), Minchenko and Stakhovski (2011 b), Pang (1997), Robinson (1976a,b), Ruszczynski (2006)), a property closely related to (first-order) constraint qualifications and to the analysis of error bounds in mathematical programming. Consider the right-hand side perturbation of the constraint set K_1, i. e. the set

$$K_1(q, r) = \{x \in \mathbb{R}^n : g(x) \leqq q,\ h(x) = r,\quad q \in \mathbb{R}^m, r \in \mathbb{R}^p\}. \tag{7}$$

Let $\bar{x} \in K_1(0, 0)$. The system (7) is *metrically regular at* $(\bar{x}, 0, 0)$ if there exist a neighborhood V of $(\bar{x}, 0, 0)$ and a constant $c > 0$ such that

$$dist(x, K_1(q, r)) \leqq c \left(\|\max\{0, g(x) - q\}\| + \|h(x) - r\| \right),\quad \forall (x, q, r) \in V,$$

where $dist(x, Y) = \inf\limits_{y \in Y} \|x - y\|$.

For smooth constraint systems, metric regularity holds if and only if the (MFC.Q.) holds for the unperturbed set K_1. "This is an important stability property that highlights the special role of (MFC.Q.) among the other C.Q.'s" (Solodov 2010)).

5. Again on the Guignard-Gould-Tolle Constraint Qualification

In the previous section we have asserted that it was proved by Gould and Tolle (1971) that the Guignard-Gould-Tolle C.Q. is the weakest possible C.Q., in the sense that it is necessary and sufficient for the *Lagrange regularity* of the triplet (g, h, X) of problem $(\mathcal{P}_1)$. For other proofs of this property, see Bazaraa and Shetty (1976), Bazaraa, Shetty, Goode and Nashed (1976), Giorgi and Zuccotti (2012). Wolkowicz (1980) considers problem $(\mathcal{P}_0)$ where all functions are assumed to be convex on the convex set $X \subset \mathbb{R}^n$, but not necessarily differentiable. With regard to this problem the said author gives a necessary and sufficient condition for the Kuhn-Tucker theory (expressed in terms of subdifferentials) to hold, independently of the objective function, i. e. the author gives a weakest constraint qualification for this problem.

We wish to stress that the *Lagrange regularity* of the triplet (g, h, X) of problem $(\mathcal{P}_1)$ is intended with regard to *all* differentiable functions having at $x^0 \in K_1$ a local minimum.

In other words, when the said Guignard-Gould-Tolle constraint qualification *does not hold* at $x^0 \in K_1$, we cannot make any conclusion about whether the (KKT) conditions hold at x^0 or not. It is therefore possible to find examples where the Guignard-Gould-Tolle C.Q. is not satisfied at a local optimal point $x^0 \in K_1$ and the related (KKT) conditions do not hold, and examples where at the local solution $x^0 \in K_1$, the (KKT) conditions hold, but the Guignard-Gould-Tolle C.Q. *does not hold.* An example of the first type is:

$$\begin{cases} \min(x_1-2)^2+(x_2+1)^2 \\ \text{s. to: } g_1(x)=(x_1-1)^3+x_2 \leqq 0, \\ g_2(x)=-x_1 \leqq 0, \\ g_3(x)=-x_2 \leqq 0. \\ x \in \mathbb{R}^2. \end{cases}$$

The optimal solution is at $x^0 = (1,0)$. At x^0 we have $I(x^0) = \{1,3\}$, $\nabla f(x^0) = (-2,2)$, $\nabla g_1(x^0) = (0,1)$, $\nabla g_2(x^0) = (-1,0)$ and $\nabla g_3(x^0) = (0,-1)$. The (KKT) conditions fail to hold at x^0 and the Guignard-Gould-Tolle C.Q. is *not* satisfied at x^0, being $T(K_0,x^0) = \{d \in \mathbb{R}^2 : d_1 \leqq 0, d_2 = 0\}$, $L(K_0,x^0) = \{d \in \mathbb{R}^2 : d_2 = 0\}$. Hence,

$$(T(K_0,x^0))^* = \{d \in \mathbb{R}^2 : d_1 \geqq 0\} \text{ and } (L(K_0,x^0))^* = \{d \in \mathbb{R}^2 : d_1 = 0\}.$$

An example of the second type is:

$$\begin{cases} \min(x_1-1)^2+(x_2+1)^2 \\ \text{s. to: } g_1(x)=(x_1-1)^3+x_2 \leqq 0, \\ g_2(x)=-x_1 \leqq 0, \\ g_3(x)=-x_2 \leqq 0. \\ x \in \mathbb{R}^2. \end{cases}$$

The optimal solution is at $x^0 = (1,0)$. The constraint qualification $(T(K_0,x^0))^* = (L(K_0,x^0))^*$ is *not* satisfied at x^0, however the (KKT) conditions hold at x^0, because $\nabla f(x^0) + 0 \cdot \nabla g_1(x^0) + 0 \cdot \nabla g_2(x^0) + 2\nabla g_3(x^0) = 0$. See Wang and Fang (2013).

Martein (1985) overcomes these "difficulties" by introducing a "regularity condition", expressed in terms of a "linearized problem". See also Giorgi (1985), Forst and Hoffmann (2010), Kleinmichel (1971). Let us consider $(\mathcal{P}_1)$, where $X \subset \mathbb{R}^n$ is open, f and every g_i, $i = 1,...,m$, are differentiable on X and every h_j, $j = 1,...,p$, is continuously differentiable on X. Let $x^0 \in K_1$; we can "linearize" the functions involved in $(\mathcal{P}_1)$ and consider the linearized problem

$$(\mathcal{P}_1)_L : \quad \begin{cases} \min \nabla f(x^0) + \nabla f(x^0)(x-x^0) \\ \text{s. to: } g_i(x^0) + \nabla g_i(x^0)(x-x^0) \leqq 0, \ i=1,...,m, \\ h_j(x^0) + \nabla h_j(x^0)(x-x^0) = 0, \ j=1,...,p. \end{cases}$$

The feasible set of $(\mathcal{P}_1)_L$ will be denoted by $(K_1)_L$. We can rewrite problem $(\mathcal{P}_1)_L$ by using the set of indices of the active inequalities at x^0, i. e.

$$(\mathcal{P}_1(I(x^0)))_L : \quad \begin{cases} \min \nabla f(x^0) + \nabla f(x^0)(x-x^0) \\ \text{s. to: } \nabla g_i(x^0)(x-x^0) \leqq 0, \ i \in I(x^0), \\ \nabla h_j(x^0)(x-x^0) = 0, \ j=1,...,p, \end{cases}$$

being of course $h_j(x^0) = 0$, $j = 1, ..., p$.

Lemma 1. Let $x^0 \in K_1$; then x^0 is a solution of $(\mathcal{P}_1)_L$ if and only if x^0 is a solution of $(\mathcal{P}_1(I(x^0)))_L$.

Proof. Let us denote by $(K_1(I(x^0)))_L$ the feasible set of $(\mathcal{P}_1(I(x^0)))_L$. If x^0 is a solution of $(\mathcal{P}_1(I(x^0)))_L$, then $\nabla f(x^0)(x - x^0) \geqq 0$ holds for $x \in (K_1(I(x^0)))_L$. Since $x^0 \in (K_1)_L \subset (K_1(I(x^0)))_L$, x^0 is also a solution of $(\mathcal{P}_1)_L$.

If conversely x^0 is a minimizer of $(\mathcal{P}_1)_L$, then $f(x^0) + \nabla f(x^0)(x - x^0) \geqq f(x^0)$, hence $\nabla f(x^0)(x - x^0) \geqq 0$ for $x \in (K_1)_L$. Let $x \in (K_1(I(x^0)))_L$; we consider the vector

$$u(t) = x^0 + t(x - x^0)$$

with $t > 0$. It holds that

$$\nabla h(x^0)(u(t) - x^0) = t\nabla h(x^0)(x - x^0) = 0.$$

For $i \in I(x^0)$:

$$\nabla g_i(x^0)(u(t) - x^0) = t\nabla g_i(x^0)(x - x^0) \leqq 0.$$

For $i \notin I(x^0)$ we have $g_i(x^0) < 0$ and thus for t sufficiently small

$$g_i(x^0) + \nabla g_i(x^0)(u(t) - x^0) = g_i(x^0) + t\nabla g_i(x^0)(x - x^0) < 0.$$

Hence, for such t the vector $u(t)$ is in $(K_1)_L$, consequently

$$f(x^0) + \nabla f(x^0)(u(t) - x^0) \geqq f(x^0)$$

and thus

$$t\nabla f(x^0)(x - x^0) = \nabla f(x^0)(u(t) - x^0) \geqq 0,$$

hence $\nabla f(x^0)(x - x^0) \geqq 0$. Since we had chosen $x \in (K_1(I(x^0)))_L$ arbitrary, x^0 yields a solution to $(\mathcal{P}_1(I(x^0)))_L$. □

Lemma 2. Let $x^0 \in K_1$; then x^0 is a solution of $(\mathcal{P}_1)_L$ if and only if $0 \in \mathbb{R}^n$ is a solution of $(\mathcal{P}_2(I(x^0)))_L$.

Proof. Obvious, on the grounds of Lemma 1. □

We remark that the feasible set of $(\mathcal{P}_2(I(x^0)))_L$ is the *linearizing cone* $L(K_1, x^0)$ of $(\mathcal{P}_1)$ at $x^0 \in K_1$. The following proposition clarifies the role of the various linearized problems with respect to the Karush-Kuhn-Tucker conditions for $(\mathcal{P}_1)$. Let $x^0 \in K_1$; the cone

$$C_d(x^0) = \left\{d \in \mathbb{R}^n : \nabla f(x^0)d < 0\right\}$$

is called the *cone of descent directions of f at x^0*.

Theorem 17. Let $x^0 \in K_1$; then the following conditions are equivalent:

(a) x^0 is a solution for $(\mathcal{P}_1)_L$;
(b) x^0 is a solution for $(\mathcal{P}_1(I(x^0))_L$;
(c) $0 \in \mathbb{R}^n$ is a solution for $(\mathcal{P}_2(I(x^0))_L$;
(d) $-\nabla f(x^0) \in (L(K_1, x^0))^*$;
(e) $L(K_1, x^0) \cap C_d(x^0) = \varnothing$;
(f) the Karush-Kuhn-Tucker conditions hold at x^0.

Proof. The previous lemmas 1 and 2 give the equivalence between (a), (b) and (c). Condition (c) means $\nabla f(x^0)d \geqq 0$ for all $d \in L(K_1, x^0)$, hence $-\nabla f(x^0) \in (L(K_1, x^0))^*$ which is condition (d). Then we note that the following equivalences hold:

$$\left\{L(K_1, x^0) \cap C_d(x^0) = \varnothing\right\} \iff \left\{\nabla f(x^0)d \geqq 0,\ \forall d \in L(K_1, x^0)\right\} \iff$$
$$\iff \left\{-\nabla f(x^0) \in (L(K_1, x^0))^*\right\}.$$

The equivalence between (e) and (f) can be easily obtained by noting that it holds:

$$\left\{d \in L(K_1, x^0) \cap C_d(x^0)\right\} \iff \begin{cases} \nabla f(x^0)d < 0 \\ \nabla g_i(x^0)d \leqq 0,\ i \in I(x^0) \\ \nabla h_j(x^0)d = 0,\ j = 1, .., p. \end{cases}$$

The last system is in turn equivalent to

$$\begin{cases} \nabla f(x^0)d < 0 \\ -\nabla g_i(x^0)d \geqq 0,\ i \in I(x^0) \\ -\nabla h_j(x^0)d \geqq 0,\ j = 1, .., p, \\ \nabla h_j(x^0)d \geqq 0,\ j = 1, .., p. \end{cases}$$

Then apply to this system, which by assumption has no solution $d \in \mathbb{R}^n$, the classical *Farkas lemma* (see Section 1), in order to obtain at once the thesis. □

We recall once more that if $x^0 \in K_1$ is a local solution of $(\mathcal{P}_1)$, it holds

$$-\nabla f(x^0) \in (T(K_1, x^0))^*,$$

being moreover $T(K_1, x^0) \subset L(K_1, x^0)$, i. e. $(L(K_1, x^0))^* \subset (T(K_1, x^0))^*$. It results therefore that if it holds

$$(L(K_1, x^0))^* = (T(K_1, x^0))^*$$

i. e. the Guignard-Gould-Tolle C.Q. holds at x^0, then the Karush-Kuhn-Tucker conditions holds at x^0.

Remark 12. Condition (a) of Theorem 17 has been called by Martein (1985) a "regularity condition" for $(\mathcal{P}_1)$, in the sense that if $x^0 \in K_1$ is a local solution of $(\mathcal{P}_1)$, then the Karush-Kuhn-Tucker conditions hold for $(\mathcal{P}_1)$ if and only if x^0 solves $(\mathcal{P}_1)_L$. This "regularity condition" is really (Theorem 17) an alternative formulation of the Karush-Kuhn-Tucker conditions and cannot be considered a true constraint qualification, to be compared with the other constraint qualifications previously introduced for $(\mathcal{P}_0)$ and $(\mathcal{P}_1)$. Martein (1985) gives also a sufficient condition for the validity of the above "regularity condition": if $x^0 \in K_1$ is a solution of $(\mathcal{P}_1)$ and x^0also solves the problem

$$\min_{x \in (K_1)_L} f(x),$$

then the Karush-Kuhn-Tucker conditions hold at x^0. The same author gives then two examples where it is shown that this last condition is independent from the Guignard-Gold-Tolle C.Q.; in the first example (example 4.1 of Martein (1985)) the Guignard-Gould-Tolle C.Q. holds at $x^0 \in K_1$, but the previous condition is not fulfilled at the same point; in the second example (example 4.2 of Martein (1985)) the previous condition is fulfilled at $x^0 \in K_1$, whereas the Guignard-Gould-Tolle C.Q. is not fulfilled at the same point.

6. Notes on Second-Order Constraint Qualifications

Second-order constraint qualifications are obviously related with second-order optimality conditions for $(\mathcal{P}_0)$ or $(\mathcal{P}_1)$. We consider $(\mathcal{P}_1)$ under the assumption that the functions involved in this problem are twice continuously differentiable on the open set $X \subset \mathbb{R}^n$.

The next theorem states the classical second-order necessary conditions for local optimality in problem $(\mathcal{P}_1)$. These conditions are essentially due to McCormick (1967, 1976, 1983); see also Fiacco and McCormick (1968), Bazaraa, Sheraly and Shetty (2006), Avriel (1976), Giorgi and Zuccotti (2007-2008).

The *Lagrangian function* for $(\mathcal{P}_1)$ is defined as

$$\mathcal{L}(x,u,w) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x), \quad u_i \geqq 0, \ i = 1, ..., m; \ w_j \in \mathbb{R}, \ j = 1, ..., p.$$

Theorem 18. Suppose x^0 is a local solution of $(\mathcal{P}_1)$ and that the (LIC.Q.) holds at x^0. Then, the (KKT) conditions hold at x^0 with associated unique multiplier vectors u and w (the set $\Lambda(x^0)$ is a singleton) and the additional Second-Order Necessary Conditions (SONC) holds at x^0 :

$$z^\top \nabla_x^2 \mathcal{L}(x,u,w) z \geqq 0$$

for all $z \in Z(x^0)$, where $Z(x^0)$ is the so-called *critical cone or cone of critical directions,*defined as follows

$$Z(x^0) = \left\{ \begin{array}{ll} z \in \mathbb{R}^n : & \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \\ & \nabla g_i(x^0) z = 0, \ i \in I(x^0) \text{ such that } u_i > 0; \\ & \nabla h_j(x^0) z = 0, \ j = 1, ..., p. \end{array} \right\}$$

Han and Mangasarian (1979) noted that if the (KKT) conditions are verified at $x^0 \in K_1$, the critical cone can equivalently be defined as

$$Z(x^0) = \left\{ \begin{array}{ll} z \in \mathbb{R}^n : & \nabla f(x^0) z = 0; \\ & \nabla g_i(x^0) z \leqq 0, \ i \in I(x^0); \\ & \nabla h_j(x^0) z = 0, \ j = 1, ..., p. \end{array} \right\}$$

We may call the above conditions, following Ben-Tal (1980), "strong second-order necessary optimality conditions" for $(\mathcal{P}_1)$.

It is known (see Arutyunov (1991), Anitescu (2000), Baccari (2004)) that the *Mangasarian-Fromovitz C.Q.* is *not* a second-order constraint qualification which assures the thesis of Theorem 18. We shall see that the *Strict Mangasarian-Fromovitz C.Q.* (necessary and sufficient for $\Lambda(x^0)$ to be a singleton) is a second-order C.Q. for the strong second-order necessary optimality conditions for $(\mathcal{P}_1)$. However, the validity of (MFC.Q.), as well as of *any* first-order constraint qualification, allows to obtain weaker forms of second-order necessary optimality conditions for $(\mathcal{P}_1)$. We need the following result.

Lemma 3. Let $x^0 \in S \subset \mathbb{R}^n$ be a local solution of the problem

$$\min_{x \in S} f(x),$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice continuously differentiable on the open set X containing S. Let $\nabla f(x^0) = 0$. Then it holds

$$y^\top \nabla^2 f(x^0) y \geqq 0, \quad \forall y \in T(S, x^0).$$

Proof. Let $y \neq 0$ be any vector of $T(S, x^0)$ and, without loss of generality, let us assume that $\|y\| = 1$. Then, there will exist a sequence $\{x^k\} \subset S$, $x^k \neq x^0$, $x^k \longrightarrow x^0$, such that

$$\lim_{k \longrightarrow \infty} \frac{x^k - x^0}{\|x^k - x^0\|} = y.$$

Then, by assumption, the quotients

$$\frac{f(x^k) - f(x^0)}{\|x^k - x^0\|^2} = \frac{\frac{1}{2}(x^k - x^0)^\top \nabla^2 f(x^0)(x^k - x^0) + o(\left\|x^k - x^0\right\|^2)}{\|x^k - x^0\|^2}$$

for k large enough are nonnegative, being x^0 a local minimum point of f on S, and converge to $\frac{1}{2} y^\top \nabla^2 f(x^0) y$. □

It is possible to prove that if S is a convex polyhedral set, the above lemma can be formulated in a weaker version: $y \in T(S, x^0)$, $\nabla f(x^0) y = 0 \Longrightarrow y^\top \nabla^2 f(x^0) y \geqq 0$.

Let us now consider $(\mathcal{P}_1)$, let $x^0 \in K_1$ be a local solution of $(\mathcal{P}_1)$ and assume that some constraint qualification is satisfied at x^0 (for example, the Guignard-Gould-Tolle C.Q., which is the most general C.Q.). Then, at x^0 the Karush-Kuhn-Tucker conditions hold. Let be

$$I^+(x^0) = \left\{ i \in I(x^0) : u_i > 0 \right\} \subset I(x^0).$$

Let be

$$K_1^+ = \left\{ x \in K_1 : g_i(x) = 0, \ \forall i \in I^+(x^0) \right\}.$$

We denote by $\mathcal{L}(x,u,w)$ the usual Lagrangian function for $(\mathcal{P}_1)$:

$$\mathcal{L}(x,u,w) = f(x) + ug(x) + wh(x).$$

Thanks to the complementarity conditions we have

$$\mathcal{L}(x,u,w) = f(x), \quad \forall x \in K_1^+.$$

As x^0 is a local solution of $(\mathcal{P}_1)$, the same point is also a local solution of the problem

$$\min_{x \in K_1^+} \mathcal{L}(x,u,w) = f(x).$$

But, thanks to the Karush-Kuhn-Tucker conditions, it holds $\nabla_x \mathcal{L}(x^0,u,w) = 0$. Applying Lemma 3, we have the following result.

Theorem 19. Let $x^0 \in K_1$ be a local solution of $(\mathcal{P}_1)$ and let (x^0,u,w) a triplet satisfying the Karush-Kuhn-Tucker conditions. Then we have

$$y^\top \nabla_x^2 \mathcal{L}(x^0,u,w) y \geqq 0, \ \forall y \in T(K_1^+, x^0).$$

Obviously if it holds

$$Z(x^0) = T(K_1^+, x^0), \tag{8}$$

then Theorem 19 may be viewed as a relaxed version of Theorem 18. Indeed, it is possible to prove that the *Strict Mangasarian-Fromovitz C.Q.* (SMFC.Q.) is a sufficient condition to obtain (8). We recall that (SMFC.Q.) is also a first-order C.Q. and that it holds

$$\boxed{\text{LIC.Q.}} \Longrightarrow \boxed{\text{SMFC.Q.}}.$$

Therefore (LIC.Q.) assures the validity of (SMFC.Q.), which in turn assures the validity of relation (8): we obtain at once the thesis of Theorem 18. Hence, similarly to the (LIC.Q.), the (SMFC.Q.) is both a first-order and a second-order C.Q. for the strong second-order necessary optimality conditions. This was observed also by Kyparisis (1985).

Another way to obtain strong necessary second-order optimality conditions for $(\mathcal{P}_1)$, with the use of any first-order C.Q. (and hence also of the (MFC.Q.)), is presented by McCormick (1967) in his pioneering paper on second-order optimality conditions for a mathematical programming problem. Let us define the cone

$$Z_1(x^0) = \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0) z = 0, \ i \in I(x^0); \ \nabla h_j(x^0) = 0, \ j = 1, ..., p \right\}$$

which is in general smaller than the critical cone $Z(x^0)$. The two cones coincide when the *strict complementarity condition* holds at x^0, i. e. in the Karush-Kuhn-Tucker conditions it holds $u_i > 0$ for every $i \in I(x^0)$. The strict complementarity condition plays an important role in many questions of mathematical programming, especially in sensitivity and stability analysis. See, e. g., Fiacco (1983), McCormick (1983).

- The *McCormick Second-Order C.Q.* is said to hold at $x^0 \in K_1$ if every nonzero $z \in Z_1(x^0)$ is tangent to a twice differentiable arc, contained in the boundary of K_1; that is, for each $z \in Z_1(x^0)$, $z \neq 0$, there exists a twice differentiable function α defined on $[0, \epsilon) \subset \mathbb{R}$ and with range in $\mathbb{R}^n$, such that $\alpha(0) = x^0$,

$$g_i(\alpha(\theta)) = 0, \quad i \in I(x^0); \;\; h_j(\alpha(\theta)) = 0, \;\; j = 1, ..., p,$$

for $0 \leqq \theta \leqq \epsilon$ and

$$\frac{d\alpha(0)}{d\theta} = \lambda z,$$

for some positive number λ.

We have the following classical result.

Theorem 20 (McCormick). Let $x^0 \in K_1$ be a local solution of $(\mathcal{P}_1)$ and suppose that there exist vectors $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ such that the Karush-Kuhn-Tucker conditions are satisfied at x^0. Further suppose that the McCormick Second-Order C.Q. holds at x^0. Then, for every $z \in Z_1(x^0)$ we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0.$$

It must be noted that the (LIC.Q.) implies the McCormick Second-Order C.Q.. So, also for this case, the (LIC.Q.) is both a first-order C.Q. and a second-order C.Q.. However, the (MFC.Q.) does not imply the McCormick Second-Order C.Q.. Fiacco and McCormick (1968) remark that the Kuhn-Tucker (first-order) C.Q. does not imply the McCormick S.O.C.Q. (indeed (KTC.Q.) is weaker than (MFC.Q.)). Also the viceversa does not hold, i. e.

$$\boxed{\text{McCormick S.O.C.Q.}} \;\; \not\Rightarrow \;\; \boxed{\text{KTC.Q.}}.$$

Consider, for example, the point $x^0 = (0, 0)$ belonging to the feasible set defined by the following constraints:

$$\begin{aligned} g_1 &= (x_1)^2 + (x_2)^2 - 2x_2 \leqq 0, \\ g_2 &= (x_1)^2 + (x_2)^2 + 2x_2 \leqq 0, \\ g_3 &= -x_1 \leqq 0. \end{aligned}$$

The McCormick S.O.C.Q. is trivially satisfied, as there does not exist $z \neq 0$ orthogonal to the gradients $(0, -2), (0, 2), (-1, 0)$. On the other hand the (KTC.Q,) is not satisfied at $(0, 0)$.

Finally, we note that if g and h are linear affine functions, the McCormick Second-Order C.Q. is automatically satisfied.

A third way to use the (MFC.Q.) in obtaining second-order necessary optimality conditions for $(\mathcal{P}_1)$ is offered by Ben-Tal (1980) in his basic paper on second-order optimality conditions. See also Still and Streng (1996). Ben-Tal introduces the so-called "weak second-order optimality conditions", in the sense that his optimality conditions "do not speak about the existence of *fixed multipliers,* but rather about the existence of multipliers which are *functions of the critical directions*" (Ben-Tal (1980)).

- *Ben-Tal Second-Order C.Q.* Let $x^0 \in K_1$. The gradients $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent and there exist some critical direction $z \in Z(x^0)$ and a vector $h \neq 0$, $h \in \mathbb{R}^n$, such that

$$\nabla g_i(x^0)h + z^\top \nabla^2 g_i(x^0)z < 0,$$

for $i \in I(x^0, z)$ and

$$\nabla h_j(x^0)h + z^\top \nabla^2 h_j(x^0)z = 0,$$

for $j = 1, ..., p$, where $i \in I(x^0, z)$ if and only if $i \in I(x^0)$ and $\nabla g_i(x^0)z = 0$.

A similar C.Q. plays a role also in infinite-dimensional settings; see Ben-Tal and Zowe (1982).

Ben-Tal (1980) shows that the above condition is a first-order constraint qualification and that the *(MFC.Q.) implies the above condition.* However, the Ben-Tal Second-Order C.Q. *does not guarantee that the same multipliers vector* (u, w) *can be chosen to validate his second-order optimality conditions.* More formally, we have the following result ("weak second-order necessary optimality conditions for $(\mathcal{P}_1)$". We recall that we have denoted by $\Lambda(x^0)$ the set of Karush-Kuhn-Tucker multipliers at the local minimum $x^0 \in K_1$.

Theorem 21 (Ben-Tal). Let $x^0 \in K_1$ be a local solution of $(\mathcal{P}_1)$. Assume that at x^0 the (MFC.Q.) or even the Ben-Tal S.O.C.Q. holds. Then, the Karush-Kuhn-Tucker conditions hold at x^0 (i. e. $\Lambda(x^0) \neq \varnothing$) and for every $z \in Z(x^0)$ there exists $(u, w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z \geqq 0. \tag{9}$$

An example of an optimization problem where, at a minimizer x^0, different multipliers are really necessary to get (9) for different $z \in Z(x^0)$, is to be found in Ben-Tal (1980), Example 2.1.

The *Constant Rank C.Q.* (CRC.Q.), introduced by Janin (1984) - see the previous section - was recognized by Andreani, Echagüe and Schuverdt (2010) to be also a second-order constraint qualification. Indeed, these authors obtain, under (CRC.Q.), strong second-order optimality conditions.

Theorem 22. Let $x^0 \in K_1$ be a local solution for $(\mathcal{P}_1)$ and let the (CRC.Q,) condition be satisfied at x^0. Then $\Lambda(x^0) \neq \varnothing$ and for every pair $(u, w) \in \Lambda(x^0)$ it holds

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u.w)z \geqq 0, \quad \forall z \in Z(x^0).$$

In the same paper the above quoted authors establish another necessary second-order optimality condition under the *weak constant rank condition:*

- Let $x^0 \in K_1$; we say that at x^0 the *weak constant rank condition* (WCR) holds if there is a neighborhood $U(x^0)$ such that the matrix made of the gradients

$$\left\{\nabla g_i(x^0),\ i \in I(x^0)\right\} \cup \left\{\nabla h_j(x^0),\ j = 1, ..., p\right\}$$

has the same rank for all $x \in U(x^0)$.

The (WCR) condition was introduced by Andreani and others (2007), where it is proved that this condition is *not* a first-order C.Q.. The above definition is different from the weak constant rank condition introduced by Liu (1995) and used by the same author in the study of sensitivity properties for nonlinear programming problems and variational inequalities.

Theorem 23. Let $x^0 \in K_1$ be a local solution for $(\mathcal{P}_1)$ and let $\Lambda(x^0) \neq \varnothing$ (i. e. the Karush-Kuhn-Tucker conditions are verified at x^0). Let the (WCR) condition be satisfied at x^0. Then, for any pair $(u, w) \in \Lambda(x^0)$ it holds

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u.w) z \geqq 0, \quad \forall z \in Z_1(x^0). \tag{10}$$

Hence, any constraint qualification which ensures $\Lambda(x^0) \neq \varnothing$ in combination with the (WCR) condition guarantees that (10) holds at a local minimizer x^0 of $(\mathcal{P}_1)$. For example, under (RCRC.Q.) we have both $\Lambda(x^0) \neq \varnothing$ and the validity of the weak constant rank condition, hence relation (10) holds at a local minimizer x^0 of $(\mathcal{P}_1)$.

Except the Ben-Tal constraint qualification and the McCormick constraint qualification, all other constraint qualifications discussed in the present section were either first-order constraint qualifications or were based on at most first-order information about the constraint qualifications. There are in the literature second-order optimality conditions, for scalar and vector programming problems, expressed in terms of *second-order tangent sets,* in a direction $d \in \mathbb{R}^n$. Here we do not treat these subjects: see, e. g., Bonnans and Shapiro (2000), Bonnans, Cominetti and Shapiro (1999), Cominetti (1990), Cambini and Martein (2002), Cambini, Martein and Vlach (1999), Kawasaki (1998), Jiménez and Novo (2004), Giorgi, Jiménez and Novo (2010), Penot (1998), Ruszczynski (2006).

Another approach to optimality conditions and to constraint qualifications for the case, e. g. in $(\mathcal{P}_1)$, where no constraint qualification is satisfied at $x^0 \in K_1$, has been suggested by various Russian mathematicians; see, e. g., Arutyunov (2000), Avakov (1989), Arutyunov, Avakov and Izmailov (2008), Avakov, Arutyunov and Izmailov (2007), Izmailov (1999), Izmailov and Solodov (2001), Brezhneva and Tret'yakov (2017).

7. Some Historical Remarks and Conclusions

We have already remarked that the notion and the term "constraint qualification" were introduced by H. W. Kuhn and A. W. Tucker (1951) in their basic paper on nonlinear programming. Obviously, under the name "regularity conditions", in the "classical" constrained optimization problems (i. e. optimization problems with equality constraints only), conditions assuring the existence of Lagrange multipliers at an optimal point were known since the beginning of this theory. The same is true also for the classical variational problems ("calculus of variations"). An interesting paper on constraint qualifications, containing also historical comments, is due to Hurwicz and Richter (2003). We shall follow mainly the approach of these authors. For other considerations on history of nonlinear programming the reader is referred to Giorgi and Kjeldsen (2014) and to the references quoted in this book.

The "classical" constrained optimization problems (with equality constraints only) are treated with a rather modern approach in the book of Hancock (1917). Hancock is perhaps the first author to suggest the possibility of handling constrained optimization problems with inequality constraints by converting them into problems with equality constraints (Hancock, page 150). A constraint $g_i(x) \leqq 0$ would be replaced by the equality $g_i(x) + (z_i)^2 = 0$.

Valentine (1937) uses the same conversion device in treating a calculus of variations minimization problem with inequality side conditions.

The Master of Science dissertation of W. Karush (Karush (1939)) went unnoticed for many years although it contains most of the basic concepts and many of the results of later works on modern nonlinear programming problems. In particular, it contains the constraint qualification that will be used by Kuhn and Tucker (1951) in their basic paper on nonlinear programming. Karush introduces also second-order optimality conditions for a nonlinear programming problem.

The paper of F. John (1948) is perhaps the first printed work concerning mathematical programming problems with inequality constraints. However, as it is well-known, this author is not concerned with the question of constraint qualifications. This question was explicitly treated by Kuhn and Tucker (1951) who, at that time, were unaware of Karush's work: see Kuhn (1976) and Giorgi and Kjeldsen (2014).

The first paper entirely concerned with constraint qualifications for a mathematical programming problem is due to Arrow, Hurwicz and Uzawa (1961). Good reference books on linear and nonlinear programming problems, appeared at the end of the fifties of the last century, are the collection edited by Arrow, Hurwicz and Uzawa (1958) and the book of Karlin (1959). One of the first author to use the Bouligand tangent cone in nonlinear programming problems is perhaps Hestenes (1966), however the relations of this cone with the linearizing cone is focused by Abadie (1967). Also Varaiya (1967) applies the Bouligand tangent cone to the analysis of optimality conditions for mathematical programming problems.

In the present paper we have given an up-to-date overview of the various constraint qualifications proposed in literature for a mathematical programming problem. The analysis has been performed under first- and second-order differentiability assumptions. The various relationships existing among the constraint qualifications considered have been pointed out. A suggestion for further developments of this subject could be to try to complete and bring up-to-date the existing works concerning the case of nonsmooth problems, i. e. problems whose functions admit some (generalized) directional derivative; in particular for the case of locally Lipschitz functions admitting Clarke generalized derivatives.

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